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A note on graphs with two kinds of zero forcing number

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***Abstract*— Zero forcing number and positive semidefinite zero forcing number are important parameters in studying minimum rank problems. Several graphs are proposed, which have the same zero forcing number and positive semidefinite zero forcing number in this paper.**

***Index Terms*—zero forcing number, positive semidefinite zero forcing number, minimum rank.**

I. INTRODUCTION

Given an n -by- n Hermitian matrix A , its graph G denoted by $A(G)$, is the undirected simple graph on vertices corresponding to the row or column indices of A , in which there is an edge between i and j if and only if $a_{ij} \neq 0, i \neq j$. For a given graph G its associated matrices set is defined as

$$\varphi(G)=\left\{A|A(G)=G,A=A^{*}\right\}$$

The minimum rank of graph G and the positive semidefinite minimum rank of G are defined as follows, respectively,

$$mr(G)=\min\{rank(A):A\in\phi(G)\}$$

and

$$msr(G)=\min\{rank(A):A\in\varphi(G) \text{ and } A \text{ is positive semidifinite}\}.$$

For G the maximum nullity and the maximum positive semidefinite nullity are, respectively,

$$M(G)=\max\{null(A):A\in\varphi(G)\}$$

and

$$M_{+}(G)=\max\{null(A):A\in\varphi(G) \text{ and } A \text{ is positive semidifinite}\}$$

Obviously

$$mr(G)+M(G)=|G|,$$

$$M(G)\geq M_{+}(G)$$

$$mr(G)\leq msr(G),$$

and

$$msr(G)+M_{+}(G)=|G|,$$

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Let $G=(V_G,E_G)$ be a simple undirected graph and $W\subseteq V_G$. The induced subgraph of W is the subgraph formed by all the edges in G between the vertices of W , denoted by $G[W]$. The subgraph induced by V_G-W is always denoted by $G-W$. If $W=\{v\}$, denote it by $G-v$.

The zero forcing sets and zero forcing number were initially introduced in [1] to provide an lower bound of $mr(G)$, and then they were extended to calculate the $msr(G)$. Given a coloring of graph G , in which each vertex is colored white or black, denote the set of black vertices by S . The derived set of S is the set of black vertices obtained by applying color change rule (positive semidefinite color change rule) until no more changes occur. The zero forcing set (positive semidefinite zero forcing set) is a set Z of black vertices whose derived set is V_G.

Given a coloring of G and we have a black vertices set $S\subseteq V(G)$. Starting from vertices in S , the rules are defined as follows:

- Color change rule: If vertex u is black and only one of its neighbor v is white, then change the color of v black. We say u forces v and write $u\rightarrow v$.

- Positive semidefinite color change rule: Denote by $\{W_1,W_2,\cdots,W_k\}$, the components of $G-S$. If $v\in W_i$, $u\in S$, and v is the white neighbor of u in $G[W_i\cup S]$, then change the color of v to black. We say u forces v and write $u\rightarrow v$.

The zero forcing number (positive semidefinite zero forcing number) denote by

$Z(G)(Z_{+}(G))$, is defined as the minimum size of all zero forcing set (positive semidef-inite zero forcing set). We have

$$Z(G)=\min\{|Z|:Z \text{ is a zero forcing set of } G\}$$

and

$$Z_{+}(G)=\min\{|Z|:Z \text{ is a positive semidefinite zero forcing set of } G.\}$$

A (positive semidefinite) zero forcing set Z is called a minimum (positive semidefinite) zero forcing set if

$Z = Z(G)(Z = Z_+(G))$.

Example 1.1. See the Figure 1, $Z_1 = \{2,3,4\}$ an zero forcing set and $Z_2 = \{1,2,4\}$ is an positive semidefinite zero forcing set. Note that Z_2 is also an zero forcing set.

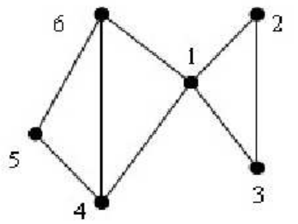

Figure 1: Example 1.1

Actually, in order to getting an positive semidefinite zero forcing set, we just decompose the graph into several certain subgraphs, and on each part using zero forcing rule independently.

Theorem 1.2[1]. For any graph G,

$$M(G) \le Z(G) \text{ and } mr(G) \ge |G| + Z(G).$$

Theorem 1.3[2]. For any graph G,

$$M_+(G) \le Z_+(G) \text{ and } msr|G| \ge |G| - Z_+(G).$$

Observation 1.4[2]. Since an zero forcing set is a positive semidefinite zero forcing set, we have

$$Z_+(G) \le Z(G).$$

For a tree τ, it is well known that $M(\tau) \le Z(\tau)$ and $M_+(\tau) \le Z_+(\tau)$. Only path P_n among trees satisfies $Z_+(G) = Z(G)$, so for path $msr(G) = mr(G)$ holds. Apart from path, we know the complete graph also satisfies the above equation. In Section 2 we give a certain graph G such that $Z_+(G) = Z(G)$, and discuss several graphs which have the same minimum rank and positive semidefinite minimum rank.

II. MAIN RESULTS

A vertex v is said to be a cut vertex of graph G, if $G - v$ is disconnected. The vertex connectivity $k(G)$ is defined as the minimum size of set $W \subseteq G$ such that $G - W$ is disconnected or a single vertex, and W is called cut set. A maximal connected induced subgraph without a cut vertex is called a block.

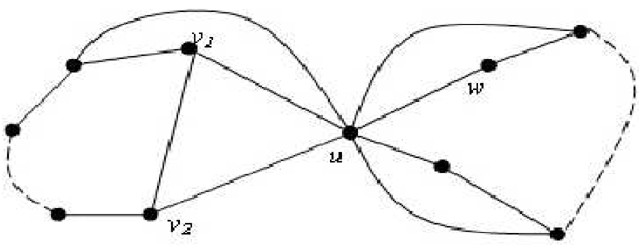

Figure 2: Graph for Theorem 2.1 (b).

Theorem 2.1. Let G is a simple graph of order n with $k(G) = 1$ and only one cut vertex u which splits G into two components, and denote $G - u = H_1 \cup H_2$. We have $Z_+(G) = Z(G)$ if one of the following conditions holds

$(a)\ G = P_3;$

$(b)\ H_i \cup u \text{ is a cycle}, i = 1,2;$

$(c)\ H_1 \text{ is a cycle and } H_2 \text{ is a path};$

$(d)\ H_i \text{ is a cycle}, i = 1,2.$

Proof. (a): It is obvious.

(b): See the graph in Figure 2. We prove that every positive semidefinite minimum zero forcing set Z of G contains exactly three vertices. For each graph $H_i \cup u$,(i=1,2) at least two black vertices are needed. So

$$|Z| \ge 3.$$

On the other side, the set $\{u, v, w\}$ is a positive semidefinite zero forcing set. We have

$$|Z| \ge 3,$$

then

$$Z_+(G) = 3.$$

By observation, we know $\{u, v, w\}$ is also a zero forcing set. Consider Observation 1.4., $Z_+(G) = Z(G)$ holds.

(c):See the graph in Figure 3. Label the graphs induced by $H_1 \cup \{u\}$ as G_1, G_2 ,respectively.Let set Z be a minimum positive semidefinite forcing set of G, then

$$Z_+(G) = |Z|.$$

By Observation 1.4. we have

$$Z_+(G) \le Z(G).$$

The remaining is to prove the converse holds, too. We claim that Z is a zero forcing set of G.

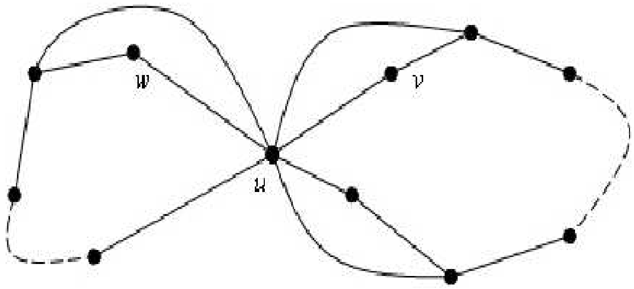

Figure 3: Graph for Theorem 2.1 (c).

Case I: $u \notin Z$. The two color change rule is same for this graph. The vertices in Z are all on graph H_1 or H_2. If the vertices of $G - u$ are all adjacent to vertex u, the set of black vertices is not a cut set when applying positive semidefinite color change rule. If not, there may be a set $\{v_1, v_4\}$ satisfying $v_1 \sim u$ and $v_4 \sim u$ and a path exists between them. Replace v_1 by v_5 in Z and denote the new set by Z_1. Do this until no such set exists, then the new set Z_1 is an zero forcing set of G, and

$$|Z_1| = |Z| = Z_+(G).$$

So

$$Z(G) \leq Z_+(G)$$

holds.

Case II: If $u \in Z$, applying positive semidefinite color change rule is the same

$$Z(G) \leq Z_+(G).$$

As using color change rule in G_1 and G_2, independently. The set $Z \cap G_1(Z \cap G_2)$ is a zero forcing set of $G_1(G_2)$. If u do not force any vertices, easy to know

$$Z(G) \leq Z_+(G)$$

If not, denote the maximum zero forcing chain starting from u in G_1 is $u \rightarrow \cdots \rightarrow v$. The set after replacing u with v in Z is a zero forcing set of G. So

$$Z(G) \leq Z_+(G)$$

holds.

In summary, $Z(G) = Z_+(G)$ holds.

(d): See the graph in Figure 4. We can prove it by similar method as (3).

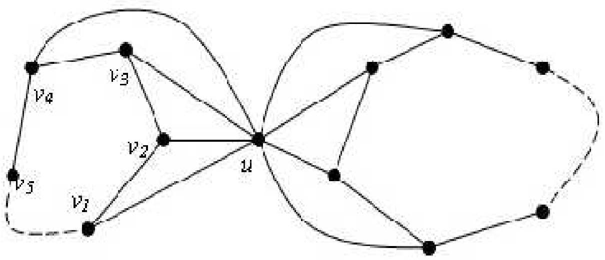

Figure 4: Graph for Theorem 2.1 (d).

Theorem 2.2. For a graph G, if $M_+(G) = Z_+(G) = Z(G)$, then $M(G) = Z(G)$.

Proof. According to

$$M_+(G) \leq M(G) \leq Z(G) = Z_+(G) = M_+(G),$$

the conclusion holds. 2

Lemma 2.3 [3]. Let G be a connected graphs with blocks $G_1, \cdots, G_m$. Then

$$msr(G) = \sum_{i=1}^{m} msr(G_i)$$

The Cartesian product of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if

(1) $u = u'$ and $v \sim v'$,

or

(2) $v = v'$ and $u \sim u'$.

Lemma 2.4[4]. For $s \geq 2$,

$$M_+(P_s \square P_2) = Z_+(P_s \square P_2) = 2$$

and

$$msr(P_s \square P_2) = 2s - 2.$$

Lemma 2.5[4]. For $s \geq 4$,

$$M_+(C_8 \square P_2) = Z_+(C_8 \square P_2) = 4$$

and

$$msr(C_8 \square P_2) = 2s - 4.$$

A graph G is said to be a superposition of two graphs G_1 and G_2 if G is obtained by identifying G_1 and G_2 at a set of vertices, keeping all the edges that are present in either G_1 or G_2.

Proposition 2.6. Let G be the superposition at one vertex of $P_s \square P_2$ and $P_t \square P_2$ with $s \geq 2, t \geq 2$. Then $Z_+(G) = Z(G) = M(G) = 3$.

Proof. See Figure 5. According to Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} msr(G) &= msr(P_s \square P_2) + msr(P_t \square P_2) \\ &= 2s - 2 + 2t - 2 \\ &= 2(s+t) - 4. \end{aligned}$$

Then

$$\begin{aligned} M_+(G) &= 2(s+t) - 1 - msr(G) \\ &= 2(s+t) - 1 - (2(s+t) - 4) \\ &= 3. \end{aligned}$$

By Theorem 1.3 and Observation 1.4,

$$3 = M_+(G) \leq Z_+(G) \leq Z(G).$$

Besides, the set $\{u, v, w\}$ is an zero forcing set of G. So

$$Z_+(G) = Z(G) = 3.$$

Since Theorem 2.2, we know the conclusion holds. Similarly, we can prove the following results are true.

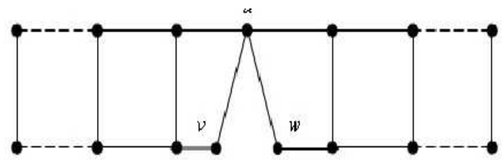

Figure 5: superposition of $P_s \square P_2$ and $P_t \square P_2$ on vertex u.

Proposition 2.7. Let G be the superposition at one vertex of $C_s \square P_2$ and $C_t \square P_2$ with $s \geq 4, t \geq 4$. Then

$$Z_+(G) = Z(G) = M(G) = 7.$$

Proposition 2.8. Let G be the superposition at one vertex of $P_s \square P_2$ and $C_t \square P_2$ with $s \geq 2, t \geq 4$. Then

$$Z_+(G) = Z(G) = M(G) = 5.$$

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