On The Existence of A Unique Solution for Systems of Ordinary Differential Equations of First Order

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***Abstract*— In this paper, we state and prove a theorem for local existence of a unique solution for a system of nonlinear ordinary differential equations (NODE) of first order by proving that the nonlinear operator of this system is contractive in a metric space subset of the Banach space consisting of the bounded differentiable functions on the real numbers and equipped with a Bielescki's type norm. Finally, we give examples to illustrate our result.**

***Index Terms*— Banach space of bounded functions $X(t) \in C'(\mathbb{R})$, Existence of *a* unique solution globally, System of nonlinear ordinary differential equations of first order.**

I. INTRODUCTION

In 2015, Bojeldain [1] proved a theorem for the existence of a unique solution for nonlinear ordinary differential equations of order m.

In this paper we study the system of nonlinear ordinary differential equations of first order having the general form:

$$X'(t) = F(t, X(t)), \qquad (1)$$

with the initial condition,

$$X(a) = C, \qquad (2)$$

where $t \geq a$ is a finite real number, and

$$X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} \qquad (3)$$

$$X(a) = \begin{bmatrix} x_1(a) \\ x_2(a) \\ x_3(a) \\ \vdots \\ x_n(a) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \qquad (4)$$

$$F(t, X(t)) = \begin{bmatrix} f_1(t, X(t)) \\ f_2(t, X(t)) \\ f_3(t, X(t)) \\ \vdots \\ f_n(t, X(t)) \end{bmatrix} \qquad (5)$$

In other form the system is

$$x_i'(t) = f_i(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t))$$

for $i = 1,2,3,\cdots,n$.

Since the system of (NODE) (1) with the initial condition (2) is equivalent to the integral equation:

$$X(t) = C + \int_a^t F(\tau, X(\tau))d\tau \qquad (7)$$

we denote the right hand side (r.h.s.) of (7) by the nonlinear operator $Q(X)t$; then prove that this operator is contractive in a metric space E subset of the Banach space B of the class of bounded functions $X(t) \in C'(\mathbb{R})$ defined by:

$$B = \{(t, X(t)) \,|\, |t-a| < \infty, |x_i(t) - c_i| \leq T_i' < \infty, \; i = 1,2,3,\cdots,n\} \qquad (8)$$

and equipped with the weighted norm:

$$\|X\| = \max_{|t-a|<\infty} (exp(-L|t-a|) \textstyle\sum_{i=1}^n |x_i(t)|) \qquad (9)$$

which is known as Bielescki's type norm [2], $L = max(l, 1)$ is a finite real number where $l = max(l_i)$, l_i is the Lipschitz coefficient of $f_i(t, X(t))$ for $i = 1,2,3,\cdots,n$ in $B1$ (a subset of the Banach space B given by (8)) defined by:

$$B1 = \{(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t)) \,|\, |t-a| \leq T, |x_i(t) - c_i| \leq T_i \leq T_i'\} \qquad (10)$$

where

T and T_i for $i = 1,2,\ldots,n$ are finite real numbers.

When the function F in the r.h.s of (1) depends linearly on its arguments except t, then equation (1) is a 1^{st} order system of linear ordinary differential equations and to prove the existence of a unique solution for it in $[a-T, a+T]$ one usually prove that component wise in a neighbourhood $N_\delta(a)$ for $t \in [a, a+\delta]$, then mimic the same steps of the proof for $t \in [a-\delta, a]$; after that use another theorem to show whether the solution do exist for all $t \in [a-T, a+T]$ or not as in [3]. By the theorem which we are going to state and prove in this paper one can easily prove the existence of a unique solution for 1^{st} order nonlinear systems of ordinary differential equations on the general form (1) for all $t \in [a-\delta, a+\delta]$ directly in a very simple metric space $\mathbb{E}$ consisting of the functions $X(t) \in C'[a-T, a+T]$, subset of the Banach space (8) [4], and equipped with the simple efficient norm (9) for $|t-a| \leq \delta$, moreover if the Lipschitz condition (11) is guaranteed to be satisfied in the Banach space (8), then the theorem guarantees the existence of a unique solution for $|t-a| < \infty$ in most cases and not in general as mentioned in [5].

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Note that this theorem is valid for 1^{st} order linear systems of ordinary differential equations as well.

II. MAIN THEOREM

Now, we are in a position to state and prove our main theorem.

Theorem: Consider the system of (NODE) (1) with the initial condition (2) and suppose that the function F in the r.h.s. of (1) is continuous and satisfies the Lipschitz condition:

$$|F(t,X(t))-F(t,Y(t))| \le l|X(t)-Y(t)| = \\ = l\sum_{i=1}^{n}|x_i - y_i| \qquad (11)$$

in $B1$ given by (10); then the initial value problem (1) and (2) has a unique solution in the $(n+1)$ dimensional metric space E (of the functions $X(t) \in C'[a-\delta, a+\delta]) \subseteq B$ defined by:

$$E = \{(t, x_1(t), x_2(t), x_3(t), \cdots, x_n(t))| \, |t-a| \le \delta, |x_i(t) - c_i| \le T^*\} \qquad (12)$$

such that $\delta = min\,(T, \frac{T^*}{M})$; where $T^* = min(T_i)$, $M = max(M_i)$, and $|f_i(t, X(t))| \le M_i$, for $i = 1,2,3,\cdots,n$ in $B1$

.

Proof: Integrating both sides of (1) from a to t and using the initial condition(2), we obtain the system of integral equations(7).

To form a fixed point problem $X(t) = Q(X)t$ denote the r.h.s. of (7) by $Q(X)t$, and to apply the contraction mapping theorem we first show that $Q: E \to E$; then prove that Q is contractive in E.

We see that:

$$. \quad |Q(X)t - C| = \left|\int_a^t F(\tau, X(\tau))d\tau\right| \le \\ \le \int_a^t |F(\tau, X(\tau))|d\tau \le \\ \le \int_a^t M\, d\tau \le M|t-a| \le M\delta \le M\frac{T^*}{M} \le T^* \qquad (13)$$

hich means that $Q: E \to E$.

Next we prove that Q is contractive, to do so we consider the difference:

$$|Q(X)t - Q(Y)t| = |Q(X) - Q(Y)|(t) = \\ = \left|\int_a^t \left(F(\tau, X(\tau)) - F(\tau, Y(\tau))\right) d\tau\right| \le \\ \le \int_a^t |F(\tau, X(\tau)) - F(\tau, Y(\tau))| d\tau \qquad (14)$$

which according to Lipschitz condition (11) yields:

$$|Q(X) - Q(Y)|(t) \le l \int_a^t |X(\tau) - Y(\tau)| d\tau \le \\ \le L\int_a^t \sum_{i=1}^{n}|x_i(\tau) - y_i(\tau)|d\tau \qquad (15)$$

Multiplying the most r. h. s. of (15) by $exp(-L|\tau - a|)\, exp(L|\tau - a|)$, we get

$$|Q(X) - Q(Y)|(t) \le \\ L\int_a^t \left(\sum_{i=1}^{n} |x_i(\tau) - y_i(\tau)|\, exp(-L|\tau - a|)\right) exp(L|\tau - a|)\, d\tau \le L\int_a^t \max_{|t-a|\le\delta}\left(exp(-L|\tau - a|) \sum_{i=1}^{n} |x_i(\tau) - y_i(\tau)|\right) exp(L|\tau - a|)\, d\tau \qquad (16)$$

which is (according to (9)),

$$|Q(X) - Q(Y)|(t) \le L\|X - Y\| \int_a^t exp(L|\tau - a|)\, d\tau = \\ = \|X - Y\|(exp(L|t-a|) - 1) \qquad (17)$$

i.e.,

$$|Q(X) - Q(Y)|(t) \le \|X - Y\|(exp(L|t-a|) - 1)$$

(18)

Multiplying both sides of (18) by $exp(-L|t-a|)$ leads to:

$$exp(-L|t-a|)|Q(X) - Q(Y)|(t) \le \|X - Y\| \cdot \\ (1 - exp(-L|t-a|)) \le \|X - Y\|(1 - exp(-L\delta)) \qquad (19)$$

The most r. h. s. of (19) is independent of t, thus it is an upper bound for its l. h. s. for any $|t-a| \le \delta$; whence:

$$\max_{|t-a|\le\delta}\left(exp(-L|t-a|)|Q(X) - Q(Y)|(t)\right) \le \\ \le \|X - Y\|(1 - exp(-L\delta)) \qquad (20)$$

which, according to the norm definition (9), gives:

$$\|Q(X) - Q(Y)\| \le (1 - exp(-L\delta))\|X - Y\| \qquad (21)$$

Since $0 < (1 - exp(-L\delta)) < 1$; then $Q(X)t$ is a contraction operator in E and has a unique solution for $t \in N_\delta(a)$.

III. EXAMPLS

In this section, we give two examples illustrate the above obtained result.

Example 3.1 We selected the exact solutions:

$$\left.\begin{aligned} x_1^*(t) &= t^2 \\ x_2^*(t) &= e^t \\ x_3^*(t) &= t+4 \end{aligned}\right\}, \qquad (22)$$

and constructed the following system of nonlinear ordinary differential equations:

$$\left.\begin{aligned} x_1'(t) &= 2t - x_1^2 + 2t^2x_1 - t^4 \\ x_2'(t) &= 2e^t - x_2 \\ x_3'(t) &= -x_3^2 + 2x_3t + 8x_3 - t^2 - 8t - 15 \end{aligned}\right\} \qquad (23)$$

If we a= 0 in (22), we get

$$\left.\begin{aligned} x_1^*(0) &= 0 \\ x_2^*(0) &= 1 \\ x_3^*(0) &= 4 \end{aligned}\right\}, \qquad (24)$$

as the initial conditions to (23).

Selecting positive finite real numbers T_1, T_2, and T_3 we find that $|x_i - c_i| \le T_i$ leads to $|x_1(t)| \le T_1, |x_2(t)| \le T_2 + 1$, $|x_3(t)| \le T_3 + 4$.

The subset $B1$ is:

$$\{(t,x_1(t),x_2(t))|\ |t-a|\le T, |x_1(t)|\le T_1,\ |x_2(t)|\le T_2+1, |x_3(t)|\le T_3+4\ \} \quad (25)$$

In $B1$ we have:

$$|f_1(t,x_1(t),x_2(t),x_3(t))| = |2t - x_1^2 + 2t^2x_1 - t^4| \le$$

$$\le 2T + T_1^2 + 2T^2T_1^2 + T^4,$$

$$|f_2(t,x_1(t),x_2(t),x_3(t))| = |2e^t - x_2| \le 2e^T + T_2 + 1,$$

and

$$|f_3(t,x_1(t),x_2(t),x_3(t))| =$$
$$= |-x_3^2 + 2x_3t + 8x_3 - t^2 - 8t - 15| \le$$
$$\le (T_3+4)(T_3+2T+12) + T(T+8) + 15$$

i.e. $M_1 = 2T + T_1^2 + 2T^2T_1^2 + T^4$, $M_2 = 2e^T + T_2 + 1$, *and* $M_3 = (T_3+4)(T_3+2T+12) + T(T+8) + 15$.

Next, we check the Lipschitz condition for $f_1, f_2, and\ f_3$:

$$|f_1(t,X(t) - f_1(t,Y(t))| = |-x_1^2 + 2t^2x_1 + y_1^2 - 2t^2y_1| \le$$
$$\le 2(T_1 + T^2)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|) \quad (26)$$

therefore f_1 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_1 = 2(T_1 + T^2)$,

$$|f_2(t,X(t)) - f_2(t,Y(t))| = |2e^t - x_2 - 2e^t - y_2| \le$$
$$\le |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|, \quad (27)$$

i.e. f_2 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_2 = 1$,

and

$$|f_3(t,X(t)) - f_3(t,Y(t))| =$$
$$= |-x_3^2 + 2x_3t + 8x_3 + y_3^2 - 2y_3t - 8y_3| \le$$
$$\le 2(8 + T_3 + T)(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|); \quad (28)$$

whence f_3 satisfies the Lipschitz condition (11) in $B1$ given by (25) with Lipschitz coefficient $l_3 = 2(8 + T_3 + T)$.

Therefore $L = max(2(T_1 + T^2), 1, 2(8 + T_3 + T))$.

Putting $M = max(M_1, M_2, M_3) = k_2T\ and\ T^* = min(T_1, T_2, T_3) = k_1T$ such that the k_1, k_2 are positive real numbers, we find that the unique solution exists in the interval $|t-a| \le \delta$ where,

$$\delta = \begin{cases} T, & if\ k_1 \le k_2 \\ \frac{k_1}{k_2}T, & if\ k_2 > k_1 \end{cases}.$$

Example 3.2 [5] As a second example, consider the following system of nonlinear ordinary differential equations:

$$\left.\begin{array}{r} x_1' + t^2\cos x_1 + x_2 = 0 \\ x_2' + \sin x_1 = 0 \end{array}\right\}, \quad (29)$$

having the initial conditions:

$$\left.\begin{array}{r} x_1(a) = 0 \\ x_2(a) = 1 \end{array}\right\}. \quad (30)$$

Selecting positive finite real numbers T_1, T_2 we find that $|x_i - c_i| \le T_i$ leads to $|x_1(t)| \le T_1$, $|x_2(t)| \le T_2 + 1$.

The subset $B1$ is:

$$\{(t,x_1(t),x_2(t))|\ |t-a|\le T, |x_1(t)|\le T_1,\ |x_2(t)|\le T_2+1\} \quad (31)$$

in which,

$$|f_1(t,x_1(t),x_2(t))| = |-t^2\cos x_1 - x_2| \le t^2(|\cos x_1|) + +|x_2| \le (T+a)^2 + T_2 + 1$$

and $|f_2(t,x_1(t),x_2(t))| = |-\sin x_1| \le 1$, i.e. $M_1 = (T+a)^2 + T_2 + 1$, $M_2 = 1$. Hence $M = max(M_1, M_2) = (T+a)^2 + T_2 + 1$.

Next, we check the Lipschitz condition for f_1 and f_2:

$$|f_1(t,X(t)) - f_1(t,Y(t))| =$$
$$= |-t^2\cos x_1 - x_2 + t^2\cos y_1 + y_2| \le$$
$$\le t^2|\cos x_1 - \cos y_1| + |x_2 - y_2| \le$$
$$\le 2(T+a)^2(|x_1 - y_1| + |x_2 - y_2|), \quad (32)$$

and

$$|f_2(t,X(t)) - f_2(t,Y(t))| = |-\sin x_1 + \sin y_1| \le$$
$$\le |x_1 - y_1| \le (|x_1 - y_1| + |x_2 - y_2|) \quad (33)$$

therefore f_1 and f_2 satisfie the Lipschitz condition (11) in $B1$ given by (31) with Lipschitz coefficient $l_1 = 2(T+a)^2$ and $l_2 = 1$ respectively.

Hence $L = max(l, 1) = 2(T+a)^2$.

Putting $M = k_2T$, $T^* = k_1T$ such that the k_1, k_2 are positive real numbers, we find that the unique solution exists in the interval $|t-a| \le \delta$ where,

$$\delta = \begin{cases} T, & if\ k_1 \le k_2 \\ \frac{k_1}{k_2}T, & if\ k_2 > k_1 \end{cases}$$

IV. CONCLUSION

We see that the contraction coefficient $0 < (1 - exp(-L\delta)) < 1$ for any finite $\delta > 0$. Moreover, in most cases, if the function F in the r.h.s. of (1) is continuous and satisfies Lipschitz condition in the Banach space (8) with finite positive Lipschitz coefficient, then the theorem is proved for t in any interval I of finite length because the contraction coefficient $(1 - exp(-L\mu(I)))$ will be positive and less than 1; where $\mu(I)$ is the measure of the interval I.

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