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Complex fuzzy linear systems†

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***Abstract*— In paper the complex fuzzy linear equation $C\tilde{z}=\tilde{w}$ in which C is a crisp complex matrix and $\tilde{w}$ is an arbitrary complex fuzzy numbers vector, is investigated. The complex fuzzy linear system is converted to a equivalent high order fuzzy linear system $G\tilde{x}=\tilde{b}$. Numerical procedure for calculating the complex fuzzy solution is designed and the sufficient condition for the existence of strong complex fuzzy solution is derived. A example is given to illustrate the proposed method.**

***Index Terms*—Complex fuzzy numbers; Matrix analysis; Complex fuzzy linear systems; Fuzzy approximate solutions.**

I. INTRODUCTION

Since many real-world engineering systems are too complex to be defined in precise terms, imprecision is often involved in any engineering design process. Fuzzy systems have an essential role in this fuzzy modeling, which can formulate uncertainty in actual environment. In many linear systems, some of the system parameters are vague or imprecise, and fuzzy mathematics is a better tool than crisp mathematics for modeling these problems, and hence solving a fuzzy linear system is becoming more important. The concept of fuzzy numbers and arithmetic operations with these numbers were firstly introduced and investigated by Zadeh [28], Dubois and Prade [11] and Nahmias [18]. A different approach to fuzzy numbers and the structure of fuzzy number spaces was given by Puri and Ralescu [20], Goetschell and Voxman [13] and Wu and Ma Ming [24; 25].

In 1998, Friedman et al.[12] proposed a general model for solving an $n\times n$ fuzzy linear systems by an embedding approach. In the past decades, a lot of results about how to deal with some advanced fuzzy linear systems such as dual fuzzy linear systems (DFLS), general fuzzy linear systems (GFLS), full fuzzy linear systems (FFLS), dual full fuzzy linear systems (DFFLS) and general dual fuzzy linear systems (GDFLS) have been set forth. These works can be seen in [1--6; 10; 23; 26]. Some fuzzy matrix system also have been studied by the same way. In 2011, Gong et al. [14] investigated a class of fuzzy matrix equations $A\tilde{X}=\tilde{B}$ by the same way. In 2013, Guo et al. [15; 16] proposed a computing method for solving fuzzy Sylvester matrix equations $A\tilde{X}+\tilde{X}B=\tilde{C}$.

For complex fuzzy system of linear equations, few researchers have developed methods to solve them in the past decades. The fuzzy complex numbers was introduced firstly by J.J. Buckley[8] in 1989. In 2010, M.A. Jahanigh[17] studied firstly the $n\times n$ fuzzy complex linear systems.

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Solution of fuzzy complex linear system of linear equations was described and was applied to circuit analysis problem by Rahgooy et al.[21]. In 2014, Behera and Chakraverty [6] discussed the fuzzy complex system of linear equations by the embedding method and redefine the complex fuzzy number. In this paper the complex fuzzy linear system $C\tilde{z}=\tilde{w}$ is investigated uniformly. A numerical procedure for calculating the fuzzy solution is designed and a sufficient condition for the existence of strong fuzzy solution is derived based on the right-hand side elements expressed by different complex fuzzy numbers recently. Finally, a example is given to illustrate our method.

II. PRELIMINARIES

There are several definitions for the concept of fuzzy numbers (see [13; 18; 27; 28]).

Definition 2.1. A fuzzy number is a fuzzy set like $u:R\to I=[0,1]$ which satisfies:

(1) u is upper semicontinuous,

(2) u is fuzzy convex, i.e., $u(\lambda x+(1-\lambda)y)\geq \min\{u(x),u(y)\}$ for all $x,y\in R,\lambda\in[0,1]$,

(3) u is normal, i.e., there exists $x_0\in R$ such that $u(x_0)=1$,

(4) $\sup pu=\{x\in R, |u(x)>0\}$ is the support of the u, and its closure cl(suppu) is compact.

Let E^1 be the set of all fuzzy numbers on R.

Definition 2.2. A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r),\overline{u}(r), 0\leq r\leq 1$, which satisfies the requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

(3) $\underline{u}(r)\leq\overline{u}(r), 0\leq r\leq 1$.

A crisp number x is simply represented by $(\underline{u}(r),\overline{u}(r))=(x,x)$, $0\leq r\leq 1$. By appropriate definitions the fuzzy number space $\{(\underline{u}(r),\overline{u}(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space.

Definition 2.3.

Let $x=(\underline{x}(r),\overline{x}(r)), y=(\underline{y}(r),\overline{y}(r))\in E^1, 0\leq r\leq 1$ and $k\in R$. Then

(1) $x=y$ *iff* $\underline{x}(r)=\underline{y}(r), \overline{x}(r)=\overline{y}(r)$,

(2) $x+y=\left(\underline{x}(r)+\underline{y}(r),\overline{x}(r)+\overline{y}(r)\right)$,

(3) $x-y=\left(\underline{x}(r)-\overline{y}(r),\overline{x}(r)-\underline{y}(r)\right)$,

(4) $kx=\begin{cases}\left(k\underline{x}(r),k\overline{x}(r)\right), k\ge 0,\\ \left(k\overline{x}(r),k\underline{x}(r)\right), k<0.\end{cases}$

Definition 2.4. An arbitrary complex fuzzy number should be represented as $\tilde{x}=\tilde{p}+i\tilde{q}$, where $\tilde{p}=\left(\underline{p}(r),\overline{p}(r)\right)$ and $\tilde{q}=\left(\underline{q}(r),\overline{q}(r)\right)$, for all $0\le r\le 1$. In this case, $\tilde{x}$ can be written as $\tilde{x}=\left(\underline{p}(r),\overline{p}(r)\right)+i\left(\underline{q}(r),\overline{q}(r)\right)$.

Definition 2.5. For any two arbitrary complex fuzzy numbers $\tilde{x}=\tilde{p}+i\tilde{q}$ and $\tilde{y}=\tilde{u}+i\tilde{v}$ where $\tilde{p},\tilde{q},\tilde{u},\tilde{v}$ are fuzzy numbers, their arithmetic is as follows:

(1) $\tilde{x}+\tilde{y}=(\tilde{p}+\tilde{u})+i(\tilde{q}+\tilde{v})$,

(2) $k\tilde{x}=k\tilde{p}+ik\tilde{q}, k\in R$,

(3) $\tilde{x}\times\tilde{y}=(\tilde{p}\times\tilde{u}-\tilde{q}\times\tilde{v})+i(\tilde{p}\times\tilde{v}+\tilde{q}\times\tilde{u})$.

Definition 2.6. The linear system equation

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ c_{n1} & c_{n2} & \cdots & c_{nn}\end{pmatrix}\begin{pmatrix}\tilde{z}_1\\ \tilde{z}_2\\ \vdots\\ \tilde{z}_n\end{pmatrix}=\begin{pmatrix}\tilde{w}_1\\ \tilde{w}_2\\ \vdots\\ \tilde{w}_n\end{pmatrix}, \tag{2.1}$$

where $c_{ij}, 1\le i,j\le n$ are complex numbers and $\tilde{w}_{ij}, 1\le i,j\le n$ are complex fuzzy numbers, is called a complex fuzzy linear system(CFLS).

Using matrix notation, we have

$$C\tilde{z}=\tilde{w}, \tag{2.2}$$

A complex fuzzy numbers vector

$$\tilde{z}=(\tilde{z}_1,\tilde{z}_2,\cdots\tilde{z}_n)^T$$

is called a fuzzy solution of the complex fuzzy linear system (2.1) if $\tilde{z}$ satisfies (2.2).

III. Solving Complex Fuzzy Linear System

Definition 3.1. An arbitrary complex fuzzy vector should be represented as $\tilde{x}=\tilde{p}+i\tilde{q}$, where $\tilde{p}=\left(\underline{p}(r),\overline{p}(r)\right)$ and $\tilde{q}=\left(\underline{q}(r),\overline{q}(r)\right)$, for all $0\le r\le 1$ are two fuzzy number vectors. In this case, the complex fuzzy vector $\tilde{x}$ can be written as $\tilde{x}=\left(\underline{p}(r),\overline{p}(r)\right)+i\left(\underline{q}(r),\overline{q}(r)\right)$.

Theorem 3.1. The $n\times n$ complex fuzzy linear system (2.1)is equivalent to a $2n\times 2n$ order fuzzy linear system

$$G\tilde{x}=\tilde{b}, \tag{3.1}$$

where

$$G=\begin{pmatrix} A & -B\\ B & A\end{pmatrix}, \tilde{x}=\begin{pmatrix}\tilde{p}\\ \tilde{q}\end{pmatrix}, \tilde{b}=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}. \tag{3.2}$$

Proof. We denote $C=A+iB, A,B\in R^{n\times n}$ and $\tilde{w}=\tilde{u}+i\tilde{v}$ where $\tilde{u}$ and $\tilde{v}$ are fuzzy number vectors. We also suppose the unknown vector $\tilde{z}=\tilde{p}+i\tilde{q}$ where $\tilde{p}$ and $\tilde{q}$ are two unknown fuzzy number vectors.

Since $C\tilde{z}=\tilde{w}$, we have

$$(A+iB)(\tilde{p}+i\tilde{q})=\tilde{u}+i\tilde{v}$$

i.e.,

$$(A\tilde{p}-B\tilde{q})+i(A\tilde{q}+B\tilde{p})=\tilde{u}+i\tilde{v}.$$

Comparing with the coefficient of i, we have

$$\begin{cases}A\tilde{p}-B\tilde{q}=\tilde{u},\\ A\tilde{q}+B\tilde{p}=\tilde{v}.\end{cases}$$

i.e.

$$\begin{pmatrix} A & -B\\ B & A\end{pmatrix}\begin{pmatrix}\tilde{p}\\ \tilde{q}\end{pmatrix}=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}, \tag{3.3}$$

it admits a $2n$ order fuzzy linear system.

We express it in matrix form as follow

$$G\tilde{x}=\tilde{b}. \ \square$$

In order to solve the complex fuzzy linear system (2.1), we need to solve the fuzzy system of linear equations (3.1).

Firstly, we set up a computing model for solving CFLS. Then we define the complex fuzzy solution of CFLS and obtain its solution representation by means of generalized inverses of matrices. Finally, we give a sufficient condition for strong fuzzy approximate solution to the complex fuzzy linear system.

When $\tilde{z}$ and $\tilde{w}$ of fuzzy linear equation (3.1) are denoted by the parametric form i.e.

$$\tilde{w}=\left[\tilde{w}_1,\tilde{w}_2,\cdots,\tilde{w}_n\right]^T,$$

$$\tilde{w}_j=\tilde{u}_j+i\tilde{v}_j=\left[\underline{u}_j(r),\overline{u}_j(r)\right]+i\left[\underline{v}_j(r),\overline{v}_j(r)\right],$$

$$\tilde{z}=\left[\tilde{z}_1,\tilde{z}_2,\cdots\tilde{z}_n\right]^T,$$

$$\tilde{z}_j=\tilde{p}_j+i\tilde{q}_j=\left[\underline{p}_j(r),\overline{p}_j(r)\right]+i\left[\underline{q}_j(r),\overline{q}_j(r)\right],$$

$$j=1,2,\cdots,n, 0\le r\le 1,$$

some results for solving fuzzy linear equation (3.1) are obtained by the following analysis.

Theorem 3.2. The fuzzy linear equation (3.1) can be extended to a crisp function linear system as follows

$$SX(r)=Y(r), \tag{3.4}$$

where

$$S=\begin{pmatrix} G^+ & -G^-\\ -G^- & G^+\end{pmatrix}, X(r)=\begin{pmatrix}\underline{x}(r)\\ \overline{x}(r)\end{pmatrix},$$

$$Y(r)=\begin{pmatrix}\underline{b}(r)\\ \overline{b}(r)\end{pmatrix}, \tag{3.5}$$

in which the elements g^+_{ij} of matrix G^+ and g^-_{ij} of matrix G^+ are determined by the following way:

if $g_{ij}\ge 0, g^+_{ij}=g_{ij}$ else $g^+_{ij}=0, 1\le i,j\le n$;

if $g_{ij}<0, g^-_{ij}=a_{ij}$ else $g^-_{ij}=0, 1\le i,j\le n$.

Proof. Let $\tilde{b}=\left[\underline{b}(r),\overline{b}(r)\right] 0\le r\le 1$ and $\tilde{x}=\left[\underline{x}(r),\overline{x}(r)\right]$. Suppose $G=G^++G^-$ in which the elements g^+_{ij} of matrix G^+ and g^-_{ij} of matrix G^+ are determined by the following way:

if $g_{ij}\ge 0, g^+_{ij}=g_{ij}$ else $g^+_{ij}=0, 1\le i,j\le n$;

if $g_{ij}<0, g^-_{ij}=a_{ij}$ else $g^-_{ij}=0, 1\le i,j\le n$.

The fuzzy linear equation $G\tilde{x}=\tilde{b}$ can be expressed as

$$\left(G^{+}+G^{-}\right)\left[\underline{x}(r),\overline{x}(r)\right]=\left[\underline{b}(r),\overline{b}(r)\right] \quad (3.6)$$

Since

$$k\tilde{x}_j=\begin{cases}\left(k\underline{x}_j(r),k\overline{x}_j(r)\right), k\geq 0,\\ \left(k\overline{x}_j(r),k\underline{x}_j(r)\right), k<0\end{cases},$$

we have

$$G\tilde{x}=\begin{cases}\left(G\underline{x}(r),G\overline{x}(r)\right), G\geq 0,\\ \left(G\overline{x}(r),G\underline{x}(r)\right), G<0.\end{cases}$$

So the Eqs.(3.3) be rewritten as

$$G^{+}\left[\underline{x}(r),\overline{x}(r)\right]+G^{-}\left[\underline{x}(r),\overline{x}(r)\right]$$
$$=\left[G^{+}\underline{x}(r),G^{+}\overline{x}(r)\right]+\left[G^{-}\underline{x}(r),G^{-}\overline{x}(r)\right]$$
$$\left[G^{+}\underline{x}(r)+G^{-}\overline{x}(r),G^{+}\overline{x}(r)+G^{-}\underline{x}(r)\right]=\left[\underline{b}(r),\overline{b}(r)\right]$$

Thus we get

$$\begin{cases}G^{+}\underline{x}(r)+G^{-}\overline{x}(r)=\underline{b}(r),\\ G^{+}\overline{x}(r)+G^{-}\underline{x}(r)=\overline{b}(r).\end{cases}$$

or

$$\begin{cases}G^{+}\underline{x}(r)-G^{-}\left(-\overline{x}(r)\right)=\underline{b}(r),\\ G^{+}\left(-\overline{x}(r)\right)-G^{-}\underline{x}(r)=-\overline{b}(r).\end{cases} \quad (3.7)$$

Expressing Eqs. (3.7) in matrix form, we have

$$\begin{pmatrix}G^{+} & -G^{-}\\ -G^{-} & G^{+}\end{pmatrix}\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}. \quad \square$$

From Eqs. (3.7), we have the following results to the property of analytical solutions.

Remark 3.1. if $C\tilde{z}=\tilde{w}$, then $\underline{z}+\overline{z}$ and $\underline{z}-\overline{z}$ are the solutions of complex fuzzy linear system, i.e., $C\left(\underline{z}+\overline{z}\right)=\left(\underline{w}+\overline{w}\right)$and $C\left(\underline{z}-\overline{z}\right)=\left(\underline{w}-\overline{w}\right)$.

Proof. In fact, by the computing model of $G\tilde{x}=\tilde{b}$,

$$\begin{cases}G^{+}\underline{x}(r)+G^{-}\overline{x}(r)=\underline{b}(r),\\ G^{+}\overline{x}(r)+G^{-}\underline{x}(r)=\overline{b}(r),\end{cases}$$

we have

$$\left(G^{+}+G^{-}\right)\underline{x}(r)+\left(G^{+}+G^{-}\right)\overline{x}(r)$$
$$=\left(G^{+}+G^{-}\right)\left(\underline{x}(r),\overline{x}(r)\right)=\underline{b}(r),\overline{b}(r),$$

i.e.,

$$G\left(\underline{x}+\overline{x}\right)=\underline{b}+\overline{b}.$$

It means if $G\tilde{x}=\tilde{b}$, then $\underline{x}+\overline{x}$ is the solution of fuzzy linear system, i.e., $G\left(\underline{x}+\overline{x}\right)=\underline{b}+\overline{b}$.

On the other hand, the fuzzy linear system $G\tilde{x}=\tilde{b}$ is equivalent to the complex fuzzy linear system $C\tilde{z}=\tilde{w}$. So we obtain the fact if $C\tilde{z}=\tilde{w}$, then $\underline{z}+\overline{z}$ is the solution of complex fuzzy linear system, i.e., $C\left(\underline{z}+\overline{z}\right)=\left(\underline{w}+\overline{w}\right)$. $\square$

Remark 3.2. if $C\tilde{z}=\tilde{w}$, then $\underline{z}-\overline{z}$ is the solution of complex fuzzy linear system, i.e., and $|C|\left(\underline{z}-\overline{z}\right)=\left(\underline{w}-\overline{w}\right)$, in which $|C|$ is the absolute values matrix of C.

Proof. In similar way, from Eqs. (3.7),

$$\begin{cases}G^{+}\underline{x}(r)-G^{-}\left(-\overline{x}(r)\right)=\underline{b}(r),\\ G^{+}\left(-\overline{x}(r)\right)-G^{-}\underline{x}(r)=-\overline{b}(r).\end{cases}$$

we have

$$\left(G^{+}-G^{-}\right)\underline{x}(r)-\left(G^{+}-G^{-}\right)\overline{x}(r)$$
$$=\left(G^{+}-G^{-}\right)\left(\underline{x}(r)-\overline{x}(r)\right)=\underline{b}(r)-\overline{b}(r),$$

i.e.,

$$|C|\left(\underline{x}-\overline{x}\right)=\underline{b}-\overline{b},$$

since $|G|=G^{+}-G^{-}$.

On the other hand, the fuzzy linear system $G\tilde{x}=\tilde{b}$ is equivalent to the complex fuzzy linear system $C\tilde{z}=\tilde{w}$. So the result that if $C\tilde{z}=\tilde{w}$, then $\underline{z}-\overline{z}$ is the solution of complex fuzzy linear system, i.e., $|C|\left(\underline{z}-\overline{z}\right)=\left(\underline{w}-\overline{w}\right)$ is obvious.

In order to solve the fuzzy system of linear equation (3.1), we need to consider the systems of linear equations (3.7). It seems that we have obtained the minimal solution of the fuzzy linear system (3.1) as

$$X(r)=S^{\dagger}Y(r),$$

i.e.,

$$\begin{pmatrix}\underline{x}(r)\\ \overline{x}(r)\end{pmatrix}=\begin{pmatrix}G^{+} & -G^{-}\\ -G^{-} & G^{+}\end{pmatrix}^{\dagger}\begin{pmatrix}\underline{b}(r)\\ \overline{b}(r)\end{pmatrix} \quad (3.8)$$

where $S^{\dagger}$ is the Moore-Penrose generalized inverse of matrix S.

However, the solution vector may still not be an appropriate fuzzy numbers vector. Restricting the discussion to triangular fuzzy numbers, i.e., $\underline{b}_i(r),\overline{b}_i(r), 1\leq i\leq 2n$ and consequently $\underline{x}_i(r),\overline{x}_i(r), 1\leq i\leq 2n$ are all linear functions of r, and having calculated $X(r)$ which solves (3.7), we define the fuzzy minimal solution to the fuzzy linear systems (2.1) as follows.

Definition 3.2. Let $X(r)=\left(\underline{x}_j(r),-\overline{x}_j(r)\right), 1\leq j\leq 2n$ denotes the minimal solution of (3.7). The fuzzy number vector

$$Z=\left[\underline{p}_j(r),\overline{p}_j(r)\right]+i\left[\underline{q}_j(r),\overline{q}_j(r)\right], 1\leq j\leq n$$

defined by

$$\underline{p}_j(r)=\min\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\},$$
$$\overline{p}_j(r)=\max\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\},$$
$$j=1,2,\cdots,n, 0\leq r\leq 1,$$
$$\underline{q}_j(r)=\min\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\},$$
$$\overline{q}_j(r)=\max\left\{\underline{x}_j(r),\overline{x}_j(r),\underline{x}_j(1),\overline{x}_j(1)\right\},$$
$$j=n+1,n+2,\cdots,2n, 0\leq r\leq 1, \quad (3.9)$$

is called the fuzzy minimal solution of the fuzzy linear systems (3.1). If $\underline{p}_i(r),\overline{q}_i(r), 1\leq i\leq 2n$ are all fuzzy numbers then

$$\tilde{z}=\left\{\left[\underline{p}_j(r),\overline{p}_j(r)\right]+i\left[\underline{q}_j(r),\overline{q}_j(r)\right] 1\leq j\leq n\right\}$$

is called a strong complex fuzzy minimal solution of the complex fuzzy linear systems (2.1). Otherwise, $\tilde{z}$ is called a weak complex fuzzy minimal solution.

To illustrate the expression (3.9) to be a fuzzy solution vector, we now discuss the generalized inverses of non negative matrix

$$S=\begin{pmatrix} G^+ & -G^- \\ -G^- & G^+ \end{pmatrix}$$

in a special structure.

Lemma 3.1[7]. Let

$$S=\begin{pmatrix} G^+ & -G^- \\ -G^- & G^+ \end{pmatrix},$$

then the matrix

$$S^\dagger=\frac{1}{2}\begin{pmatrix} \left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger & \left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger \\ \left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger & \left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger \end{pmatrix} \tag{3.10}$$

is the Moore-Penrose inverse of the matrix S, where $\left(G^++G^-\right)^\dagger$, $\left(G^+-G^-\right)^\dagger$ are Moore-Penrose inverses of matrices G^++G^- and G^+-G^-, respectively.

The key points to make the solution matrix being a strong fuzzy solution is that $S^\dagger Y(r)$ is fuzzy matrix, i.e., each element in which is a triangular fuzzy number. By the analysis, it is equivalent to the condition $S^\dagger\geq 0$.

Theorem 3.3. If

$$\left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger\geq 0,$$

$$\left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger\geq 0,$$

the complex fuzzy linear equation (2.1) has a strong complex fuzzy minimal solution as follows:

$$\tilde{z}=\left[\underline{p}(r),\overline{p}(r)\right]+i\left[\underline{q}(r),\overline{q}(r)\right],0\leq r\leq 1, \tag{3.11}$$

where

$$\begin{cases} \underline{x}(r)=\begin{pmatrix}\underline{p}(r)\\ \underline{q}(r)\end{pmatrix}=E\underline{b}(r)-F\overline{b}(r), \\ \overline{x}(r)=\begin{pmatrix}\overline{p}(r)\\ \overline{q}(r)\end{pmatrix}=-F\underline{b}(r)+E\overline{b}(r), \\ E=\frac{1}{2}\left(\left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger\right), \\ F=\frac{1}{2}\left(\left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger\right). \end{cases}$$

Proof. Let

$$S^\dagger=\begin{pmatrix} E & F \\ F & E\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} \left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger & \left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger \\ \left(G^+-G^-\right)^\dagger-\left(G^++G^-\right)^\dagger & \left(G^+-G^-\right)^\dagger+\left(G^++G^-\right)^\dagger \end{pmatrix}.$$

We know the condition that $S^\dagger\geq 0$ is equivalent to $E\geq 0$, $F\geq 0$.

Since $\tilde{b}=\left[\underline{b}(r),\overline{b}(r)\right]$, we know that $\underline{b}(r)$ is a bounded monotonic increasing left continuous vector function and $\overline{b}(r)$ a bounded monotonic decreasing left continuous vector function with $\underline{b}(r)\leq\overline{b}(r), 0\leq r\leq 1$.

According to Eqs. (3.8), we have

$$\begin{pmatrix}\underline{x}(r)\\ -\overline{x}(r)\end{pmatrix}=\begin{pmatrix} G^+ & -G^- \\ -G^- & G^+ \end{pmatrix}^\dagger\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix}$$

$$=\begin{pmatrix} E & F \\ F & E\end{pmatrix}\begin{pmatrix}\underline{b}(r)\\ -\overline{b}(r)\end{pmatrix},$$

i.e.,

$$\underline{x}(r)=E\underline{b}(r)-F\overline{b}(r), -\overline{x}(r)=F\underline{b}(r)-E\overline{b}(r).$$

Now that $E\geq 0$, $F\geq 0$ and $\underline{b}(r), -\overline{b}(r)$ are bounded monotonic increasing left continuous function vectors, we know that $\underline{x}(r)$ is a bounded monotonic increasing left continuous function vector and $\overline{x}(r)$ is a bounded monotonic decreasing left continuous function vector. And

$$\begin{pmatrix}\overline{p}(r)\\ \overline{q}(r)\end{pmatrix}-\begin{pmatrix}\underline{p}(r)\\ \underline{q}(r)\end{pmatrix}=E\left(\overline{b}(r)-\underline{b}(r)\right)+F\left(\overline{b}(r)-\underline{b}(r)\right)$$

$$=(E+F)\left(\overline{b}(r)-\underline{b}(r)\right)\geq 0.$$

Thus the complex fuzzy linear equation (2.6) has a strong fuzzy minimal solution.

The following Theorems give some results for such S^{-1} and $S^\dagger$ to be nonnegative. As usual, $(\cdot)^{\mathrm{T}}$ denotes the transpose of a matrix $(\cdot)$.

Theorem 3.6[19]. The inverse S^{-1} of a nonnegative matrix S is nonnegative if and only if S is a generalized permutation matrix.

Theorem 3.7[7]. Let S be an $2n\times 2m$ nonnegative matrix with rank r. Then the following assertions are equivalent:

(a) $S^\dagger\geq 0$,

(b). There exists a permutation matrix P, such that PS has the form

$$PS=\begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \\ 0\end{pmatrix},$$

where each T_i has rank 1 and the rows of T_i are orthogonal to the rows of T_j, whenever $i\neq j$, the zero matrix may be absent.

(c). $S^\dagger=\begin{pmatrix} GC^{\mathrm{T}} & GD^{\mathrm{T}} \\ GD^{\mathrm{T}} & GC^{\mathrm{T}}\end{pmatrix}$ for some positive diagonal matrix G. In this case,

$$(C+D)^\dagger=G(C+D)^{\mathrm{T}}, (C-D)^\dagger=G(C-D)^{\mathrm{T}}.$$

IV. Numerical Examples

Example 4.1. Consider the following CFSLE:

$$\begin{pmatrix} 1 & -1 \\ 1 & 3\end{pmatrix}\begin{pmatrix}\tilde{z}_1\\ \tilde{z}_2\end{pmatrix}=\begin{pmatrix}(r,2-r)+i(1+r,3-r)\\ (4+r,7-2r)+i(r-4,-1-2r)\end{pmatrix}.$$

Let $\tilde{z}_1=\tilde{p}_1+i\tilde{q}_1=\left(\underline{p_1},\overline{p_1}\right)+i\left(\underline{q_1},\overline{q_1}\right)$,

$$\tilde{z}_2 = \tilde{p}_2 + i\tilde{q}_2 = \left(\underline{p_2}, \overline{p_2}\right) + i\left(\underline{q_2}, \overline{q_2}\right).$$

Applying Theorem 3.1., the complex fuzzy linear is equivalent to following fuzzy linear system $Gx = b$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{q}_1 \\ \tilde{q}_2 \end{pmatrix} = \begin{pmatrix} (r, 2-r) \\ (4+r, 7-2r) \\ (1+r, 3-r) \\ (r-4, -1-2r) \end{pmatrix}.$$

Form Theorem 3.2., we need to solve the following function linear system

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \underline{p_1} \\ \underline{p_2} \\ \underline{q_1} \\ \underline{q_2} \\ -\overline{p_1} \\ -\overline{p_2} \\ -\overline{q_1} \\ -\overline{q_2} \end{pmatrix} = \begin{pmatrix} r \\ 4+r \\ 1+r \\ r-4 \\ r-2 \\ 2r-7 \\ r-3 \\ 2r+1 \end{pmatrix}.$$

By the Eqs. (3.8), we have

$$\begin{pmatrix} \underline{p_1} \\ \underline{p_2} \\ \underline{q_1} \\ \underline{q_2} \\ -\overline{p_1} \\ -\overline{p_2} \\ -\overline{q_1} \\ -\overline{q_2} \end{pmatrix} = \begin{pmatrix} G^+ & -G^- \\ -G^- & G^+ \end{pmatrix}^{\dagger} \begin{pmatrix} r \\ 4+r \\ 1+r \\ r-4 \\ r-2 \\ 2r-7 \\ r-3 \\ 2r+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.375 + 0.625r \\ 0.875 + 0.125r \\ 0.125 + 0.625r \\ -1.375 + 0.125r \\ -2.875 + 0.875r \\ -1.375 + 0.375r \\ -1.625 + 0.875r \\ 0.875 + 0.375r \end{pmatrix}.$$

Thus we obtain the solution of the complex fuzzy linear system as

$$\tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} (1.375 + 0.625r, 2.875 - 0.875r) \\ + i(0.125 + 0.625r, 1.625 - 0.875r), \\ (0.875 + 0.125r, 1.375 - 0.375r) \\ + i(-1.375 + 0.125r, -0.875 - 0.375r) \end{pmatrix},$$

which admits a strong complex fuzzy solution of original system.

V. CONCLUSION

In this work we presented a matrix method for solving complex fuzzy linear equation $C\tilde{z} = \tilde{w}$ where C is a crisp complex matrix and $\tilde{w}$ is an arbitrary complex fuzzy vector, respectively. Numerical procedure for calculating the model is designed and the sufficient condition for the existence of strong complex fuzzy solution is derived. In addition, some theorems are stated and proved to the property of analytical solutions. Numerical example showed that our method is effective to solve the complex fuzzy linear system.

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