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A Modified Minimal Gersgorin Disc Theorem of Matrix Eigenvalue

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***Abstract*— In this paper, a new improvement of the minimum gersgorin disc theorem is given, which uses the improvement of the minimum disc theorem to obtain a new region of eigenvalue distribution. Although the method is not precise, it is very useful in practical applications.**

***Index Terms*—Eigenvalue; Gerschgorin disc theorem; minimum disc theorem.**

I. INTRODUCTION

In scientific research, in many problems such as mathematical physics and numerical analysis can be transformed into solving linear system, how to calculate eigenvalues of matrix plays a very important role in the field of matrix analysis for a long time. For some large matrix, it is difficult to find out the exact value of eigenvalues. Hence, we can estimate approximate value of matrix eigenvalue, The disk theorem is a classical method for estimating the eigenvalues of a matrix. In addition, the minimal disc theorem is a more efficient theorem based on the theorem of the matrix. On the basis of previous studies, this paper a new improvement of the minimum gersgorin disc theorem is given, which uses the improvement of the minimum disc theorem to obtain a new region of eigenvalue distribution. Although the method is not precise, it is very useful in practical applications.

First we recall some basic Theorem that will be used in this paper.

Lemma 1 **(*Gerschgorin disc theorem [1]*).** Suppose that $A=(a_{ij})\in C^{n\times n}$, $n\in N$ and λ is an eigenvalue of A, let $\sigma(A)$ is spectrum of A, then

$$\sigma(A)\subseteq\Gamma(A)=\bigcup_{i\in N}\Gamma_i(A),$$

where

$$\Gamma_i(A)=\left\{z\in C:|z-a_{ii}|\le r_i(A)=\sum_{j\ne i}|a_{ij}|\right\}\ (\forall i\in N).$$

$\Gamma(A)$ named Gerschgorin region of A, $\Gamma_i(A)$ named number i Gerschgorin disc.

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Lemma 2 (Minimal Gerschgorin disc theorem [2,3]). Suppose that $A=(a_{ij})\in C^{n\times n}$, $X=diag(x_1,x_2,\cdots,x_n)$, $x_i>0\,(\forall i\in N)$, let

$$\Gamma_i^{r^x}(A)=\left\{z\in C:|z-a_{ii}|\le r_i^x(A)=\sum_{j\ne i}\frac{|a_{ij}|x_j}{x_i}\right\},i\in N$$

and

$$\Gamma^{r^x}(A)=\bigcup_{i\in N}\Gamma_i^{r^x}(A),\ \Gamma^R(A)=\bigcap_{X>0}\Gamma^{r^x}(A).$$

Then $\Gamma^R(A)$ is minimal Gerschgorin disc region of A.

proof. Suppose that

$$A=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix},$$

we get

$$X^{-1}AX=\begin{pmatrix}a_{11}&\frac{a_{12}x_2}{x_1}&\cdots&\frac{a_{1n}x_n}{x_1}\\\frac{a_{21}x_1}{x_2}&a_{22}&\cdots&\frac{a_{2n}x_n}{x_2}\\\vdots&\vdots&\ddots&\vdots\\\frac{a_{n1}x_1}{x_n}&\frac{a_{n2}x_2}{x_n}&\cdots&a_{nn}\end{pmatrix}.$$

Due to definition of eigenvalue $X^{-1}AX\cdot Y=\lambda\cdot Y$, $|y_i|=\max(|y_j|),\ j=1,2,\cdots,n.$ Thus

$$\sum_j\frac{a_{ij}x_j}{x_i}\cdot y_j=\lambda\cdot y_j,i=1,2,\cdots,n,$$

then

$$(\lambda-a_{ii})y_i=\sum_{j\ne i}\frac{a_{ij}x_j}{x_i}\cdot y_j,$$

$$|\lambda-a_{ii}||y_i|\le\sum_{j\ne i}\left|\frac{a_{ij}x_j}{x_i}\right|\cdot|y_j|,$$

we obtain

$$|\lambda-a_{ii}|\le\sum_{j\ne i}\left|\frac{a_{ij}x_j}{x_i}\right|.$$

For $\forall \lambda'$ of $X^{-1}AX$, $\lambda' \in \Gamma^{\gamma^x}(A)$, then $\lambda \in \Gamma^{\gamma^x}(A)$, we all the arbitrary of X , then $\lambda \in \Gamma^R(A)$.

The proof of Lemma 2 is completed.

II. MAIN RESULTS

In this section, based on the minimal Gerschgorin disc region of matrix, we state our main results.

Theorem 3. Suppose that $A=(a_{ij}) \in C^{n\times n}$, $n \in N$, then

$$\sigma(A) \subseteq D^R(A) \subseteq \Gamma^R(A),$$

where

$$V_{ij}^x(A)=\left\{ z\in C: \left(|\lambda - a_{ii}| - r_i^x(A) + \frac{|a_{ij}|x_j}{x_i}\right)\times \left(|\lambda - a_{jj}| - r_j^x(A) + \frac{|a_{ji}|x_i}{x_j}\right) \le |a_{ij}||a_{ji}| \right\}$$

$$D_i^x(A)=\bigcup_{j\neq i} V_{ij}^x(A),$$

$$D^x(A)=\bigcup_{i\in N} D_i^x(A),$$

$$D^R(A)=\bigcap_{X>0} D^x(A),$$

$$X=diag(x_1,x_2,\cdots,x_n) \quad x_i>0\,(\forall i\in N).$$

Proof. For $\forall \lambda \in \sigma(A)$, we easy to known that $\lambda \in \sigma(X^{-1}AX)$. Assuming that y is called an eigenvector of A associated with λ, $y=(y_1,y_2,\cdots,y_n)^T \in R^n$, due to definition of eigenvalue,

$$(X^{-1}AX)y=\lambda y \tag{2.1}$$

let $|y_p|$, $|y_q|$ be the first maximum value and the second maximum value of each component of eigenvector y, respectively, i.e. $\min\{|y_p|,|y_q|\} \ge |y_i|, i\neq \{p,q\}$. Then

$$|\lambda - a_{pp}|\cdot|y_p| \le \left(r_p^x(A) - \frac{|a_{pq}|x_q}{x_p}\right)\cdot|y_p| + \frac{|a_{pq}|x_q}{x_p}\cdot|y_q|,$$

same as

$$|\lambda - a_{pp}| \le \left(r_p^x(x) - \frac{|a_{pq}|x_q}{x_p}\right) + \frac{|a_{pq}|x_q}{x_p}\cdot\frac{|y_q|}{|y_p|},$$

(2.2)

in the same way,

$$|\lambda - a_{qq}|\cdot \le \left(r_q^x(A) - \frac{|a_{qp}|x_p}{x_q}\right) + \frac{|a_{qp}|x_p}{x_q}\cdot\frac{|y_p|}{|y_q|} \tag{2.3}$$

According to (2.2), (2.3), we can obtain

$$\left(|\lambda - a_{pp}| - r_p^x(A) + \frac{|a_{pq}|x_q}{x_p}\right)\times \left(|\lambda - a_{qq}| - r_q^x(A) + \frac{|a_{qp}|x_p}{x_q}\right) \le |a_{pq}||a_{qp}| \tag{2.4}$$

Due to the arbitrary of λ , y_p, y_q and (2.4), we could obtain, for $X=diag(x_1,x_2,\cdots,x_n)$, then all eigenvalues for A must be seated the union $D^R(A)$ compose with then $V_{ij}^x(A)$, in short $\sigma(A) \subseteq D^R(A)$.

In order to proof that the theorem is more effective than the original minimum disc theorem, so we need to proof $D^R(A) \subseteq \Gamma^R(A)$,

for $\forall z \in D^R(A)$ and $X>0$, $i,j\in N$, exit

$$\left(|z - a_{ii}| - r_i^x(A) + \frac{|a_{ij}|x_j}{x_i}\right)\times \left(|z - a_{jj}| - r_j^x(A) + \frac{|a_{ji}|x_i}{x_j}\right) \le |a_{ij}||a_{ji}| \tag{2.5}$$

According to Lemma 2, if $X>0$, $z \notin \Gamma^R(A)$, then

$$|z-a_{ii}| > r_i^x(A),$$

$$|z-a_{jj}| > r_j^x(A), j\neq i,$$

and

$$|z - a_{ii}| - r_i^x(A) + \frac{|a_{ij}|x_j}{x_i} > \frac{|a_{ij}|x_j}{x_i}, \tag{2.6}$$

$$|z - a_{jj}| - r_j^x(A) + \frac{|a_{ji}|x_i}{x_j} > \frac{|a_{ji}|x_i}{x_j}, \tag{2.7}$$

According to (2.6), (2.7), we can obtain

$$\left(|z - a_{ii}| - r_i^x(A) + \frac{|a_{ij}|x_j}{x_i}\right)\times \left(|z - a_{jj}| - r_j^x(A) + \frac{|a_{ji}|x_i}{x_j}\right) > |a_{ij}||a_{ji}|.$$

So we get a contradiction with the know, then $D^R(A) \subseteq \Gamma^R(A)$.

The proof of Theorem 3 is completed.

Corollary 4 If $A=A^T=(a_{ij}) \in C^{n\times n}$, $n\in N$, then

$$\sigma(A) \subseteq D^R(A) \subseteq \Gamma^R(A),$$

where

$$V_{ij}^{x}(A)=\{z\in C:|z-a_{ii}||z-a_{jj}|\le s_i^x(A)\cdot s_j^x(A)\}$$

$$D_i^{x}(A)=\bigcup_{j\neq i}V_{ij}^{x}(A),$$

$$D^{x}(A)=\bigcup_{i\in N}D_i^{x}(A),$$

$$D^{R}(A)=\bigcap_{X>0}D^{x}(A),$$

$$s_i^x(A)=\sum_{j\neq i}\frac{a_{ji}x_i}{x_j},\quad s_j^x(A)=\sum_{i\neq j}\frac{a_{ij}x_j}{x_i},$$

$$X=diag(x_1,x_2,\cdots,x_n)\quad x_i>0(\forall i\in N).$$

III. Conclusion

In many fields such as numerical analysis and mathematical physics, a number of problems come down to the study of linear system, how to calculate eigenvalues of matrix plays a very important role in the field of matrix analysis for a long time. For some large matrix, it is difficult to find out the exact value of eigenvalues. On the basis of previous studies, this paper a new improvement of the minimum gersgorin disc theorem is given, which uses the improvement of the minimum disc theorem to obtain a new region of eigenvalue distribution. Although the method is not precise, it is very useful in practical applications.

In addition, in the process of application, the main conclusions of the diagonal matrix X is particularly important, we generally choose the diagonal element between 0 and 1, the method is more effective. The selection of X technology is worth our further exploration.

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