On The System Of Double Equations $N_1 - N_2 = 4k + 2(k > 0)$ $N_1 N_2 = (2k+1)\alpha^2$

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Abstract— **This paper concerns with the problem of obtain infinitely many non-zero distinct integers N_1, N_2 such that $N_1 - N_2 = 4k + 2(k > 0)$ and $N_1 N_2 = (2k+1)\alpha^2$ where $2k+1$ is square-free. A few examples are given. Some observations among N_1, N_2 are presented.**

Index Terms— **Diophantine Problem, Integer Pairs, System of equations.**

Mathematics Subject Classification: 11D25, 11D04, 11D99

I. INTRODUCTION

Number theory, called the queen of Mathematics, is a board and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans). One of the oldest branches of mathematics itself, is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on[1-6]. Diophanitne problems were first introduced by Diophantus of Alexandira who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to Algebra. The theory of Diophantine equation is a treasure house in which the search for many hidden relation and properties among numbers from a treasure hunt. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [7,8]. Also one may refer [9-14].

In this communication, we attempt for obtaining two non-zero distinct integers N_1 and N_2 such that $N_1 - N_2 = 4k+2, N_1 N_2 = (2k+1)\alpha^2$, where $2k+1$ is square-free.

II. METHOD OF ANALYSIS

Let N_1, N_2 be any two non-zero distinct integers such that

$$N_1 - N_2 = 4k + 2(k > 0) \tag{1}$$

$$N_1 N_2 = (2k+1)\alpha^2 \tag{2}$$

Eliminating N_2 between (1) and (2), we have

$$N_1^2 - (4k+2)N_1 - (2k+1)\alpha^2 = 0 \tag{3}$$

Treating (3) as a quadratic in N_1 and solving for N_1, we have

$$N_1 = (2k+1) \pm \sqrt{(2k+1) + (2k+1)\alpha^2}$$

Taking $\alpha = (2k+1)\gamma$ in the above equation, we have

$$N_1 = (2k+1) \pm (2k+1)\sqrt{(2k+1)\gamma^2 + 1} \tag{4}$$

Let $Y^2 = (2k+1)\gamma^2 + 1$ (5)

whose general solution (γ_n, Y_n) is given by

$$Y_n = \frac{1}{2} f_n,$$

$$\gamma_n = \frac{1}{2\sqrt{2k+1}} g_n$$

where

$$f_n = \left(Y_0 + \sqrt{2k+1}\gamma_0\right)^{n+1} + \left(Y_0 - \sqrt{2k+1}\gamma_0\right)^{n+1}$$

$$g_n = \left(Y_0 + \sqrt{2k+1}\gamma_0\right)^{n+1} - \left(Y_0 - \sqrt{2k+1}\gamma_0\right)^{n+1}$$

Consider the positive sign in (4), the values of N_1 are given by

$$N_1 = N_1(k,n) = \frac{(2k+1)}{2}[f_n + 2] \tag{6}$$

and from (1), we have

$$N_2 = N_2(k,n) = \frac{(2k+1)}{2}[f_n - 2] \tag{7}$$

Then, (6) and (7) represent the required values of N_1 and N_2 satisfying (1) and (2).

A few numerical examples are given in the table below:

TABLE: NUMERICAL EXAMPLES

n	$N_1(k,n)$	$N_2(k,n)$
0	9k	3k
1	24k	18k
2	81k	75k
3	294k	288k
4	1089k	1083k

The recurrence relations satisfied by N_1 and N_2 are respectively given by

$$N_1(k,n+2) - 2Y_0 N_1(k,n+1) + N_1(k,n) = 2(1 - Y_0)(2k+1)$$

$$N_2(k,n+2) - 2Y_0 N_2(k,n+1) + N_2(k,n) = 2(Y_0 - 1)(2k+1)$$

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III. OBSERVATIONS

Employing the linear combinations among $N_1(k,n)$ and $N_2(k,n)$, one may obtain solutions for hyperbolas and parabolas.

ILLUSTRATION 1

Let

$X = \left[N_2(k,n+1) - Y_0 N_1(k,n) - (2k+1)\sqrt{2k+1}\gamma_0(1+Y_0)\right]$,

$Y = N_1(n,2k+1)$

Note that (X,Y) satisfies the parabola $(2k+1)^2\gamma_0{}^2 Y - 2X^2 = 2(2k+1)^3\gamma_0{}^2$.

ILLUSTRATION 2

Let

$X = \left[N_2(k,n+1) - Y_0 N_1(k,n) - (2k+1)\sqrt{2k+1}\gamma_0(1+Y_0)\right]$

$Y = N_1(k,n) - 2k - 1$

Note that (X,Y) satisfies the hyperbola $(2k+1)\gamma_0{}^2 Y^2 - X^2 = (2k+1)^3\gamma_0{}^2$

Replace n by 2n+1 in $f_n = \frac{2}{2k+1}N_1(k,n) - 2$

Note that $f_{2n+1} = \frac{2}{2k+1}\left[N_1(k,2n+1) - 2k - 1\right]$

$$\Rightarrow f_n{}^2 = \frac{2}{2k+1}\left[N_1(k,2n+1) - 2k - 1 + 2k + 1\right]$$

$$\Rightarrow f_n{}^2 = \frac{2}{2k+1}\left[N_1(k,2n+1)\right]$$

$\therefore \frac{12}{2k+1}\left[N_1(k,2n+1)\right]$ is a nasty number.

In a similar manner, Replace n by 2n+1 in $f_n = \frac{2}{2k+1}N_2(k,n) + 2$

Note that $f_{2n+1} = \frac{2}{2k+1}\left[N_2(k,2n+1) + 2k + 1\right]$

$$\Rightarrow f_n{}^2 = \frac{2}{2k+1}\left[N_2(k,2n+1) + 2k + 1 + 2k + 1\right]$$

$$\Rightarrow f_n{}^2 = \frac{2}{2k+1}\left[N_2(k,2n+1) + 2(2k+1)\right]$$

$\therefore \frac{12}{2k+1}\left[N_2(k,2n+1) + 2(2k+1)\right]$ is a nasty number

Replace n by 3n+2 in $f_n = \frac{2}{2k+1}N_1(k,n) - 2$

Note that $f_{3n+2} = \frac{2}{2k+1}\left[N_1(k,3n+2) - 2k - 1\right]$

$$\Rightarrow f_n{}^3 = \frac{2}{2k+1}\left[N_1(k,3n+2) + 3N_1(k,n) - 4(2k+1)\right]$$

$\therefore \frac{2}{2k+1}\left[N_1(k,3n+2) + 3N_1(k,n) - 4(2k+1)\right]$ is a cubical integer.

In a similar manner, Replacing n by 3n+2 in $f_n = \frac{2}{2k+1}N_2(k,n) + 2$

Note that $f_{3n+2} = \frac{2}{2k+1}\left[N_2(k,3n+2) + 2k + 1\right]$

$$\Rightarrow f_n{}^3 = \frac{2}{2k+1}\left[N_2(k,3n+2) + 3N_2(k,n) + 4(2k+1)\right]$$

$\therefore \frac{2}{2k+1}\left[N_2(k,3n+2) + 3N_2(k,n) + 4(2k+1)\right]$ is a cubical integer.

IV. CONCLUSION

In this paper, we have obtained infinitely many pairs of non-zero distinct integers such that their product is six times a square. Considering the positive values of N_1 and N_2 to represent the sides of a rectangle, it is observed that this problem gives infinitely many rectangles such that, the area of each rectangle is a nasty number.

As Diophantine problems are rich in variety, are may attempt for finding infinitely many pairs of integers satisfying other choices of relations among them.

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