The Nature Diagnosability of Bubble-sort Star Graphs under the PMC Model and MM* Model

Mujiangshan Wang, Yuqing Lin, Shiying Wang

***Abstract*—Many multiprocessor systems have interconnection networks as underlying topologies and an interconnection network is usually represented by a graph where nodes represent processors and links represent communication links between processors. No fault set can contain all the neighbors of any fault-free vertex in the system, which is called the nature diagnosability of the system. Diagnosability of a multiprocessor system is one important study topic. As a famous topology structure of interconnection networks, the n-dimensional bubble-sort star graph BS_n has many good properties. In this paper, we prove that the nature diagnosability of BS_n is $4n-7$ under the PMC model for $n \geq 4$, the nature diagnosability of BS_n is $4n-7$ under the MM* model for $n \geq 5$.**

***Index Terms*—Bubble-sort star graph, Diagnosability, Interconnection network.**

I. INTRODUCTION

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. Some processors may fail in the system, so processor fault identification plays an important role for reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system G is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of presented faults does not exceed t. The diagnosability $t(G)$ of G is the maximum value of t such that G is t-diagnosable. For a t-diagnosable system, Dahbura and Masson [1] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors. Several diagnosis models (e.g., Preparata, Metze, and Chien's (PMC) model [2], Barsi, Grandoni, and Maestrini's (BGM)model [3], and Maeng and Malek's (MM) model [4] have been proposed to investigate the diagnosability of multiprocessor systems. In particular, two of the proposed models, the PMC model and the MM model, are well known and widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM model, to diagnose a system, a node sends the same task to two of its neighbors, and then compares their responses. For this reason, the MM model is also said to be the comparison model. Sengupta and Dahbura [1] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. Numerous studies have been investigated under the PMC model and MM model or MM* model.

In the traditional diagnosis of a multiprocessor system, one generally assumes that any subset of processors may simultaneously fail. If all the neighbors of some node v are faulty simultaneously, it is impossible to determine whether v is faulty or fault-free. As a consequence, the diagnosability of the system is less than its minimum node degree. However, in some large-scale multiprocessor systems, we can safely assume that all neighbors of any node do not fail at the same time. Based on this assumption, in 2005, Lai et al. [5] introduced the restricted diagnosability of the system called the conditional diagnosability. They consider the situation that no fault set can contain all the neighbors of any node in the system. Since the probability that the all neighbors of a fault node fail and create faults is more to the probability that the all neighbors of a fault-free node fail and create faults in the system, we consider the situation that no fault set can contain all the neighbors of any fault-free node in the system, which is called the nature diagnosability of the system. In 2012, Peng et al. [6] proposed a measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability (which is also called the g-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least g fault-free neighbors. In [6], they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model. In [7], Wang and Han studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the MM* model. In 2016, Ren and Wang [8] gave some properties of the g-good-neighbor diagnosability of a multiprocessor system. In 2017, Wang et al. [9] studied that the 2-good-neighbor diagnosability of bubble-sort star graph networks under the PMC model and MM* model. Yuan et al. [10,11] studied that the g-good-neighbor diagnosability of the k-ary n-cube $(k \geq 3)$ under the PMC model and MM* model. As a favorable topology structure of interconnection networks, the Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n has many good properties. In [12], Wang et al. studied the 2-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model and MM* model. In 2016, Zhang et al. [13] proposed a new measure for fault diagnosis of the system, namely, the g-extra diagnosability, which restrains that every fault-free component has at least (g+1) fault-free nodes. In [13], they studied the g-extra diagnosability of the n-dimensional hypercube under the PMC model and MM* model. In 2016, Wang et al. [14] studied that the 2-extra diagnosability of the n-dimensional bubble-sort star graph under the PMC model and MM* model. In [15], Han and Wang studied that the

Mujiangshan Wang, School of Electrical Engineering and Computer Science, The University of Newcastle NSW 2308, Australia.

Yuqing Lin, School of Electrical Engineering and Computer Science, The University of Newcastle NSW 2308, Australia.

***Shiying Wang**, School of Mathematics and Information Science, Henan Normal University, Xinxiang, PR China, +86-03733326148.

g -extra diagnosability of folded hypercubes. In 2017, Wang and Yang [16] studied the 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model and MM* model. In [17], Wang et al. studied the nature diagnosability of $C\Gamma_n$ under the PMC model and MM* model and proved that the nature diagnosability of the system is less than or equal to the conditional diagnosability of the system. Therefore, the nature diagnosability of the system is nature and one important study topic. In 2016, Bai and Wang [18] studied the nature diagnosability of Möbius cubes; Hao and Wang [19] studied the nature diagnosibility of augmented k-ary n-cubes; Jirimutu and Wang [20] studied the nature diagnosability of alternating group graph networks; Ma and Wang [21] studied the nature diagnosability of crossed cubes; Zhao and Wang [22] studied the nature diagnosability of augmented 3-ary n-cubes. The star graph and the bubble-sort graph have been proved to be an important viable candidate for interconnecting a multiprocessor system. The feature of the star graph include low degree of node, small diameter, symmetry, and high degree of fault-tolerance. The diagnosabilities of the star graph under the PMC model and MM model were studied in [23,24]. Lin et al. [25] showed that the conditional diagnosability of the star graph under the comparison diagnosis model is $3n-7$. In this paper, the nature diagnosability of the n-dimensional bubble-sort star graph BS_n under the PMC model and MM* model has been studied. It is proved that the nature diagnosability of BS_n is $4n-7$ under the PMC model for $n\geq 4$, the nature diagnosability of BS_n is $4n-7$ under the MM* model for $n\geq 5$.

II. Preliminaries

In this section, some definitions and notations needed for our discussion, the bubble-sort star graph, the PMC model and MM* model are introduced.

A. *Definitions and Notations*

A multiprocessor system is modeled as an undirected simple graph $G=(V,E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. Given a nonempty vertex subset V' of V, the subgraph induced by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. We denote by $\delta(G)$ the minimum degrees of vertices of G. For any vertex v, we define the neighborhood $N_G(v)$ of v in G to be the set of vertices adjacent to v. u is called a neighbor or a neighbor vertex of v for $u\in N_G(v)$. Let $S\subseteq V$. We use $N_G(S)$ to denote the set $\cup_{v\in S}N_G(v)\setminus S$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. A graph G is said to be k -regular if for any vertex v, $d_G(v)=k$. Let G be a connected graph. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left when G is complete. A fault set $F\subseteq V$ is called a nature faulty set if $|N(v)\cap(V\setminus F)|\geq 1$ for every vertex v in $V\setminus F$. A nature cut of G is a nature faulty set F such that $G-F$ is disconnected. The minimum cardinality of nature cuts is said to be the nature connectivity of G, denoted by $\kappa^*(G)$. For graph-theoretical terminology and notation not defined here we follow [26].

B. *The PMC model and MM* model*

For the PMC model and MM* model, we follow [10].

In a system $G=(V,E)$, a faulty set $F\subseteq V$ is called a conditional faulty set if it does not contain all of neighbors of any vertex in G. A system G is conditional t -diagnosable if every two distinct conditional faulty subsets $F_1,F_2\in V$ with $|F_1|\leq t,|F_2|\leq t$ are distinguishable. The conditional diagnosability $t_c(G)$ of G is the maximum number of t such that G is conditional t -diagnosable. By [27], $t_c(G)\geq t(G)$.

Theorem 1. ([17]) For a system $G=(V,E)$, $t(G)=t_0(G)\leq t_1(G)\leq t_c(G)$.

In [17], Wang et al. proved that the nature diagnosability of the Bubble-sort graph B_n under the PMC model is $2n-3$ for $n\geq 4$. In [28], Zhou et al. proved the conditional diagnosability of B_n is $4n-11$ for $n\geq 4$ under the PMC model. Therefore, $t_1(B_n)<t_c(B_n)$ when $n\geq 5$ and $t_1(B_n)=t_c(B_n)$ when $n=4$.

C. *The bubble-sort star graph*

The bubble-sort star graph has been known as a famous topology structure of interconnection networks. In this section, its definition and some properties are introduced.

Let $[n]=\{1,2,\cdots,n\}$, and let S_n be the symmetric group on $[n]$. containing all permutations $p=p_1p_2\cdots p_n$ of $[n]$. It is well known that $\{(1i):2\leq i\leq n\}$ is a generating set for S_n. So $\{(1,i):2\leq i\leq n\}\cup\{(i,i+1):2\leq i\leq n-1\}$ is also a generating set for S_n. The n-dimensional bubble-sort star graph BS_n [29,30] is the graph with vertex set $V(BS_n) = S_n$ in which two vertices u, v are adjacent if and only if $u=v(1,i)$, $2\leq i\leq n$, or $u=v(i,i+1)$, $2\leq i\leq n-1$. It is easy to see from the definition that BS_n is a $(2n-3)$ -regular graph on $n!$ vertices.

Note that BS_n is a special Cayley graph. BS_n has the following useful properties.

Proposition 1. For any integer $n\geq 1$, BS_n is $(2n-3)$ -regular, vertex transitive.

Proposition 2. For any integer $n\geq 2$, BS_n is bipartite.

Proposition 3. For any integer $n\geq 3$, the girth of BS_n is 4.

Theorem 2. ([31]) Let H be a simple connected graph with $n=|V(H)|\geq 3$. If H^1 and H^2 are two different labelled graphs obtained by labelling H with $\{1,2,\cdots,n\}$, then $Cay(H^1,S_n)$ is isomorphic to $Cay(H^2,S_n)$.

We can partition BS_n into n subgraphs $BS_1,BS_2,\ldots,BS_n$, where every vertex $u=x_1x_2\ldots x_n\in V(BS_n)$ has a fixed integer i in the last position x_n for $i\in[n]$. It is obvious that

BS_n^i is isomorphic to BS_{n-1} for $i \in [n]$. Let $v \in V(BS_n^i)$. Then $v(1n)$ and $v(n-1,n)$ are called outside neighbors of v.

Proposition 3. ([29]) Let BS_n^i be defined as above. Then there are $2(n-2)!$ independent cross-edges between two different H_i's.

Proposition 4. ([29]) Let BS_n be the bubble-sort star graph. If two vertices u, v are adjacent, there is no common neighbor vertex of these two vertices, i.e., $|N(u) \cap N(v)| = 0$. If two vertices u, v are not adjacent, there are at most three common neighbor vertices of these two vertices, i.e., $|N(u) \cap N(v)| \le 3$.

Lemma 1. ([9]) The nature connectivity $\kappa^*(BS_4)$ of the bubble-sort star graph BS_4 is 8.

A connected graph G is super nature connected if every minimum nature cut F of $V(G)$ isolates one edge. If, in addition, $G-F$ has two components, one of which is an edge, then G is tightly $|F|$ super nature connected.

Theorem 3. ([14]) For $n \ge 5$, the bubble-sort star graph BS_n is tightly $(4n-8)$ super nature connected.

Lemma 2. Let $A = \{(1),(12)\}$. If $n \ge 4$, $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$, then $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \ge 1$, and $\delta(BS_n - F_2) \ge 1$.

Proof. By $A = \{(1),(12)\}$, we have $BS_n[A] \cong BS_2 = K_2$. Since BS_n has not 3-cycles, $|N_{BS_n}(A)| = 4n-8$. Thus from calculating, we have $|F_1| = 4n-8$, $|F_2| = |A| + |F_1| = 4n-6$.

Claim 1. For any $x \in S_n \setminus F_2$, $|N_{BS_n}(x) \cap F_2| \le 2n-4$.

Since BS_n is a bipartite graph, there is no 5-cycle $(1),(ki),x,(12)(lj),(12),(1)$ of BS_n, where $(ki),(lj) \in S \setminus (12)$. Let $u \in N_{BS_n}((1)) \setminus (12)$. If u is adjacent to x, then x is not adjacent to each of $N_{BS_n}((12)) \setminus (1)$. Since $|N_{BS_n}((1)) \setminus (12)| = 2n-4$, we have that x is adjacent to at most $(2n-4)$ vertices in F_1.

By Claim 1, $|N_{BS_n}(x) \cap F_2| \le 2n-4$ for any $x \in S_n \setminus F_2$. Therefore, $\delta(BS_n - F_2) \ge 2n-3-(2n-4) = 1$. $BS_n - F_1$ has two components $BS_n - F_2$ and BS_2. Note that $\delta(BS_2) = 1$. Therefore, $\delta(BS_n - F_1) \ge 1$.

III. THE NATURE DIAGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH UNDER THE PMC MODEL

In this section, we shall show the nature diagnosability of the bubble-sort star graph under the PMC model. Let F_1 and F_2 be two distinct subsets of V for a system $G = (V, E)$. Define the symmetric difference $F_1 \Delta F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Yuan et al. [10] presented a sufficient and necessary condition for a system to be nature t-diagnosable under the PMC model.

Theorem 4. ([10]) A system $G = (V, E)$ is nature t-diagnos -able under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$.

Lemma 3. A graph of minimum degree 1 has at least two vertices.

The proof of Lemma 3 is trivial.

Lemma 4. Let $n \ge 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the PMC model is less than or equal to $4n-7$, i.e., $t_1(BS_n) \le 4n-7$.

Proof. Let A be defined in Lemma 2, and let $F_1 = N_{BS_n}(A)$, $F_2 = A \cup N_{BS_n}(A)$. By Lemma 2, $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \ge 1$ and $\delta(BS_n - F_2) \ge 1$. Therefore, F_1 and F_2 are both nature faulty sets of BS_n with $|F_1| = 4n-8$ and $|F_2| = 4n-6$. Since $A = F_1 \Delta F_2$ and $N_{BS_n}(A) = F_1 \subset F_2$, there is no edge of BS_n between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. By Theorem 4, we can deduce that BS_n is not nature $(4n-6)$-diagnosable under the PMC model. Hence, by the definition of nature diagnosability, we conclude that the nature diagnosability of BS_n is less than $4n-6$, i.e., $t_1(BS_n) \le 4n-7$.

Lemma 5. Let $n \ge 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the PMC model is more than or equal to $4n-7$, i.e., $t_1(BS_n) \ge 4n-7$.

Proof. By the definition of nature diagnosability, it is sufficient to show that BS_n is nature $(4n-7)$-diagnosable. By Theorem 4, to prove BS_n is nature $(4n-7)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(BS_n)$ with $u \in V(BS_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \Delta F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(BS_n)$ with $|F_1| \le 4n-7$ and $|F_2| \le 4n-7$.

We prove this statement by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(BS_n)$ with $|F_1| \le 4n-7$ and $|F_2| \le 4n-7$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 4, i.e., there are no edges between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \ne \emptyset$. Suppose $V(BS_n) = F_1 \cup F_2$. By the definition of BS_n, $|F_1 \cup F_2| = |S_n| = n!$. It is obvious that $n! > 8n-14$ for $n \ge 4$. Since $n \ge 4$, we have that $n! = |V(BS_n)| = |F_1 \cup F_2| = |F_1| + + |F_2| - |F_1 \cap F_2| \le |F_1| + |F_2| \le 2(4n-7) = 8n-14$, a contradiction. Therefore, $V(BS_n) \ne F_1 \cup F_2$.

Since there are no edges between $V(BS_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$, and F_1 is a nature faulty set, $BS_n - F_1$ has two parts $BS_n - F_1 - F_2$ and $BS_n[F_2 \setminus F_1]$ (for convenience). Thus, $\delta(BS_n - F_1 - F_2) \ge 1$ and $\delta(BS_n[F_2 \setminus F_1]) \ge 1$. Similarly, $\delta(BS_n[F_1 \setminus F_2]) \ge 1$ when $F_1 \setminus F_2 \ne \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a nature faulty set.Since there are no edges between $V(BS_n - F_1 - F_2)$ and $F_1 \Delta F_2$, $F_1 \cap F_2$ is a nature cut. Since $n \ge 4$, by Theorem 3, $|F_1 \cap F_2| \ge 4n-8$. By Lemma 3, $|F_2 \setminus F_1| \ge 2$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \ge 2 + 4n-$

$8 = 4n-6$, which contradicts with that $|F_2| \le 4n-7$. So BS_n is nature $(4n-7)$-diagnosable. By the definition of $t_1(BS_n)$, $t_1(BS_n) \ge 4n-7$.

Combining Lemmas 4 and 5, we have the following theorem.

Theorem 5. Let $n \ge 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the PMC model is $4n-7$.

IV. THE NATURE DIAGNOSABILITY OF THE BUBBLE-SORT STAR GRAPH BS_n UNDER THE MM* MODEL

Before discussing the nature diagnosability of the bubble-sort star graph BS_n under the MM* model, we first give an existing result.

Theorem 6. ([1,10]) A system $G=(V,E)$ is nature t-diagnosable under the MM* model if and only if each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \le t$ and $|F_2| \le t$ satisfies one of the following conditions.

(1) There are two vertices $u, w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $F_1 \Delta F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.

Lemma 6. Let $n \ge 4$. Then the nature diagnosability of the bubble-sort star graph BS_n under the MM* model is less than or equal to $4n-7$, i.e., $t_1(BS_n) \le 4n-7$.

Proof. Let A, F_1 and F_2 be defined in Lemma 2. By Lemma 2, $|F_1| = 4n-8$, $|F_2| = 4n-6$, $\delta(BS_n - F_1) \ge 1$ and $\delta(BS_n - F_2) \ge 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \Delta F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(BS_n) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Theorem 6, and BS_n is not nature $(3n-6)$-diagnosable. Hence, $t_1(BS_n) \le 4n-7$. The proof is complete.

Lemma 7. Let $n \ge 5$. Then the nature diagnosability of the bubble-sort star graph BS_n under the MM* model is more than or equal to $4n-7$, i.e., $t_1(BS_n) \ge 4n-7$.

Proof. By the definition of nature diagnosability, it is sufficient to show that BS_n is nature $(4n-7)$-diagnosable.

By Theorem 6, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of BS_n with $|F_1| \le 4n-7$ and $|F_2| \le 4n-7$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 6. Without loss of generality, assume that $F_2 \setminus F_1 \ne \emptyset$. Similarly to the discussion on $V(BS_n) \ne F_1 \cup F_2$ in Lemma 5, we can deduce that $V(BS_n) \ne F_1 \cup F_2$. Therefore, $V(BS_n) \ne F_1 \cup F_2$.

Claim I. $BS_n - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $BS_n - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 6, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Similarly, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w when $F_1 \setminus F_2 \ne \emptyset$. Let $W \subseteq S_n \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $BS_n[S_n \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $S_n \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(2n-5)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \le 4n-7$, we have $\sum_{w \in W} |N_{BS_n[(F_1 \cap F_2) \cup W]}(w)| = |W|(2n-5) \le \sum_{v \in F_1 \cap F_2} d_{BS_n}(v) \le |F_1 \cap F_2|(2n-3) \le (|F_2|-1)(2n-3) \le (4n-8)(2n-3) = 8n^2 - 28n + 24$. It follows that $|W| \le \frac{8n^2-28n+24}{2n-5} < 4n-3$ for $n \ge 5$. Note $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \le 2(4n-7) - (2n-5) = 6n-9$. Suppose $V(H) = \emptyset$. Then $n! = |S_n| = |V(BS_n)| = |F_1 \cup F_2| + |W| < 6n-9+4n-3 = 10n-11$. This is a contradiction to $n \ge 5$. So $V(H) \ne \emptyset$.

Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Theorem 6, and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1 \Delta F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of BS_n and $\delta(BS_n - (F_1 \cap F_2)) \ge 1$, i.e., $F_1 \cap F_2$ is a nature cut of BS_n. By Theorem 3, $|F_1 \cap F_2| \ge 4n-8$. Because $|F_1| \le 4n-7$, $|F_2| \le 4n-7$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. According to Proposition 5, there are at most three common neighbors for any pair of vertices in BS_n, it follows that there are at most three isolated vertices in $BS_n - F_1 - F_2$, i.e., $|W| \le 3$.

Suppose that there is exactly one isolated vertex v in $BS_n - F_1 - F_2$.

Let v_1 and v_2 be adjacent to v. Then $N_{BS_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1) \setminus \{v\} \subseteq F_1 \cap F_2$; $N_{BS_n}(v_2) \setminus \{v\} \subseteq F_1 \cap F_2$; $[N_{BS_n}(v) \setminus \{v_1, v_2\}] \cap [N_{BS_n}(v_1) \setminus \{v\}] = \emptyset$ and $[N_{BS_n}(v) \setminus \{v_1, v_2\}] \cap [N_{BS_n}(v_2) \setminus \{v\}] = \emptyset$. By Proposition 5, $|[N_{BS_n}(v_1) \setminus \{v\}] \cap [N_{BS_n}(v_2) \setminus \{v\}]| \le 2$. Thus, $|F_1 \cap F_2| \ge |N_{BS_n}(v) \setminus \{v_1, v_2\}| + |N_{BS_n}(v_1) \setminus \{v\}| + |N_{BS_n}(v_2) \setminus \{v\}| = (2n-5) + (2n-4) + (2n-4) - 2 = 6n-15$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \ge 1 + 6n - 15 = 6n-14 > 4n-7$ $(n \ge 5)$, which contradicts $|F_2| \le 4n-7$.

Suppose that there are exactly two isolated vertices v and w in $BS_n - F_1 - F_2$.

Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{BS_n}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1) \setminus \{v, w\} \subseteq F_1 \cap F_2$, $N_{BS_n}(v_2) \setminus \{v, w\}$

$\subseteq F_1 \cap F_2$, $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_1)\setminus\{v,w\}]=\varnothing$ and $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_2)\setminus\{v,w\}]=\varnothing$. By Proposition 5, there are at most two common neighbors for any pair of vertices in BS_n. Thus, it follows that $|[N_{BS_n}(v_1)\setminus\{v,w\}]\cap[N_{BS_n}(v_2)\setminus\{v,w\}]|\le 1$. Thus, $|F_1\cap F_2|\ge|N_{BS_n}(v)\setminus\{v_1,v_2\}|+|N_{BS_n}(w)\setminus\{v_1,v_2\}|+|N_{BS_n}(v_1)\setminus\{v,w\}|+|N_{BS_n}(v_2)\setminus\{v,w\}|=(2n-5)+(2n-5)-1+(2n-5)+(2n-5)-1=8n-22$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\ge 1+8n-22=8n-21>4n-7$ $(n\ge 5)$, which contradicts $|F_2|\le 4n-7$.

Suppose that there are exactly three isolated vertices u,v and w in $BS_n-F_1-F_2$.

Let v_1 and v_2 be adjacent to u,v and w, respectively. Then $N_{BS_n}(v)\setminus\{v_1,v_2\}\subseteq F_1\cap F_2$. Since BS_n contains no triangle, it follows that $N_{BS_n}(v_1)\setminus\{u,v,w\}$ $\subseteq F_1\cap F_2$, $N_{BS_n}(v_2)\setminus\{u,v,w\}\subseteq F_1\cap F_2$, $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_1)\setminus\{u,v,w\}]=\varnothing$ and $[N_{BS_n}(v)\setminus\{v_1,v_2\}]\cap[N_{BS_n}(v_2)\setminus\{u,v,w\}]=\varnothing$. By Proposition 5, there are at most three common neighbors for any pair of vertices in BS_n. Thus, it follows that $|[N_{BS_n}(v_1)\setminus\{u,v,w\}]\cap[N_{BS_n}(v_2)\setminus\{u,v,w\}]|=0$. Thus, $|F_1\cap F_2|\ge|N_{BS_n}(u)\setminus\{v_1,v_2\}|+|N_{BS_n}(v)\setminus\{v_1,v_2\}|+|N_{BS_n}(w)\setminus\{v_1,v_2\}|+|N_{BS_n}(v_1)\setminus\{u,v,w\}|+|N_{BS_n}(v_2)\setminus\{u,v,w\}|=(2n-5)+(2n-5)+(2n-5)+(2n-6)+(2n-6)-3=10n-30$. It follows that $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\ge 1+10n-30=10n-29>4n-7$ $(n\ge 5)$, which contradicts $|F_2|\le 4n-7$.

Suppose $F_1\setminus F_2=\varnothing$. Then $F_1\subseteq F_2$. Since F_2 is a nature faulty set, $BS_n-F_2=BS_n-F_1-F_2$ has no isolated vertex. The proof of Claim I is complete.

Let $u\in V(BS_n)\setminus(F_1\cup F_2)$. By Claim I, u has at least one neighbor in $BS_n-F_1-F_2$. Since the vertex set pair (F_1,F_2) is not satisfied with any one condition in Theorem 6, by the condition (1) of Theorem 6, for any pair of adjacent vertices $u,w\in V(BS_n)\setminus(F_1\cup F_2)$, there is no vertex $v\in F_1\Delta F_2$ such that $uw\in E(BS_n)$ and $vw\in E(BS_n)$. It follows that u has no neighbor in $F_1\Delta F_2$. By the arbitrariness of u, there is no edge between $V(BS_n)\setminus(F_1\cup F_2)$ and $F_1\Delta F_2$. Since $F_2\setminus F_1\ne\varnothing$ and F_1 is a nature faulty set, $\delta_{BS_n}([F_2\setminus F_1])\ge 1$. By Lemma 3, $|F_2\setminus F_1|\ge 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(BS_n)\setminus(F_1\cup F_2)$ and $F_1\Delta F_2$, $F_1\cap F_2$ is a nature cut of BS_n. By Theorem 3, we have $|F_1\cap F_2|\ge 4n-8$. Therefore, $|F_2|=|F_2\setminus F_1|+|F_1\cap F_2|\ge 2+(4n-8)=4n-6$, which contradicts $|F_2|\le 4n-7$. Therefore, BS_n is nature $(4n-7)$-diagnosable and $t_1(BS_n)\ge 4n-7$. The proof is complete.

Combining Lemmas 6 and 7, we have the following theorem.

Theorem 7. Let $n\ge 5$. Then the nature diagnosability of the bubble-sort star graph BS_n under the MM* model is $4n-7$.

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