

p-Convex Functions in Discrete Sets

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Abstract— We study the new concept of a p-convex function and A-p-convex sets for some set A of a vector space E. These concepts may have applications in convex and non linear analysis and other topics of mathematical sciences.

Index Terms— Non convex analysis, convex and p-convex sets , A-p-convex sets, p-convex functions, discrete sets

I. INTRODUCTION

In this paper, we extend some concepts and theorems to non convex analysis . In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of f is defined to be the set of all points lying on or above its graph. We proved that, if S is a nonempty p-convex set in R^n and $f: S \rightarrow R^+$. Then f is p-convex if and only if the epigraph of f, p-epi f, is a p-convex set (Th. 3.3). As an example of a p-convex function is $f(x) = ||x||_p$ defined by a p-norm on a vector space (Example 3.1).

We also proved equivalent properties to A-p-convex sets for some fixed set A of E, (Th.3.5). Here the set B is said to be A-p-convex set if $B = A \cap C$ for some p-convex set C in E.

Now for two fixed subsets A and Q of a vector space E if an operator between two power sets,

$$\gamma = \gamma_{PQ} : P(E) \rightarrow P(A),$$

is defined by

$$\gamma(B) = p - cvx(B \cap Q) \cap A$$

for a real vector space E and for any subset A of E we proved that B is a A-p-convex if and only if $B = \gamma(B)$ for all $Q \supseteq A$, (Th. 3.5)

II. SOME PROPERTIES OF P-CONVEX SETS :

Throughout this paper we let $0 < p \leq 1$. Let us first go to the following definitions and examples

A “**p-norm**” on a vector space E over a field K is a mapping $f(x) = ||x||_p$ from E to $R^+ \cup \{0\}$ satisfying the following axioms,

1. $||x||_p = 0$ if and only if $x=0, x \in E$
2. $||tx||_p = |t|^p ||x||_p, t \in K, x \in E$
3. $||x + y||_p \leq ||x||_p + ||y||_p, x, y$ for all $x, y \in E$.

Let U be a set in a vector space E and $x, y \in U, s, t \geq 0$. The set

$$A_x^y = \{sx+ty; s^p+t^p=1\}, x \in E \\ = \{s^{1/p}x+t^{1/p}y; s+t=1\}$$

Is said to be the “closed arc segment “ joining x, y. A_x^y can also be written as

$$A_x^y = \{(1-t)^{1/p}x+t^{1/p}y; s^p+t^p=1\}.$$

A set A in a vector space is said to be “**P-convex**” if for all $x, y \in A, s, t \geq 0$, we have

$$s x + t y \in A \text{ whenever } s^p + t^p = 1$$

Equivalently, if we put $s=1-t$, we obtain

$$(1-t)^{1/p}x+t^{1/p}y \in A \text{ whenever } 0 \leq t \leq 1.$$

We note that the singleton set $A = \{x\}$ is convex but it is not p-convex.

As examples of p-convex sets is the arc segment defined above.

Also the closed unit ball in $B_{l_2}^p$, in the metric space l_2^p ,

$$B_{l_2}^p = \{x = (x_1, x_2); |x_1|^p + |x_2|^p \leq 1\}$$

Is a p-convex set.

We claim that $(1-t)^{1/p}x + t^{1/p}y \in B_{l_2}^p$, for $x, y \in B_{l_2}^p$

Note that

$$(1-t)^{1/p}(x_1, x_2) + t^{1/p}(y_1, y_2) \\ = ((1-t)^{1/p}x_1 + t^{1/p}y_1, \\ (1-t)^{1/p}x_2 + t^{1/p}y_2).$$

Since $(a+b)^2 \leq a^2 + b^2, a, b > 0$ and $1 \geq p > 0$, it follows that

$$|((1-t)^{1/p}x_1 + t^{1/p}y_1|^p + |((1-t)^{1/p}x_2 + t^{1/p}y_2|^p \\ \leq (|(1-t)^{1/p}x_1| + |t^{1/p}y_1|)^p + (|(1-t)^{1/p}x_2| + |t^{1/p}y_2|)^p \\ \leq (1-t)(|x_1|^p + |y_1|^p) + (1-t)(|x_2|^p + |y_2|^p) \\ \leq (1-t)+t \leq 1. \blacksquare$$

In what follows we show that the family of p-convex sets is closed under the operations of the sum and scalar multiplication.

Theorem 2.1 . If C_1 and C_2 are p-convex sets, then $C_1 + C_2$ is also p-convex, where

$$C_1 + C_2 = \{x_1 + x_2; x_1 \in C_1, x_2 \in C_2\}$$

Proof.

Let $x, y \in C_1 + C_2$ then

$x = x_1 + x_2$ and $y = y_1 + y_2$, for every $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$.

Now,

$$\begin{aligned} (1-t)^{1/p} x + t^{1/p} y &= (1-t)^{1/p} (x_1 + x_2) \\ &+ t^{1/p} (y_1 + y_2) \\ &= [(1-t)^{1/p} x_1 + t^{1/p} y_1] + [(1-t)^{1/p} x_2 \\ &+ t^{1/p} y_2] \in C_1 + C_2. \end{aligned}$$

Therefore $C_1 + C_2$ is a p-convex set. ■

Theorem 2.2. If C is a p-convex set, then αC is also p-convex, where

$$\alpha C = \{\alpha x, x \in C, \alpha \in K\}$$

Proof. Let $x, y \in \alpha C$, then $x = \alpha x_1$ and $y = \alpha y_1$ for every $x_1, y_1 \in C$.

Now

$$\begin{aligned} (1-t)^{1/p} x + t^{1/p} y &= (1-t)^{1/p} \alpha x_1 + \\ &+ t^{1/p} \alpha y_1 \\ &= \alpha [(1-t)^{1/p} x_1 + \\ &+ t^{1/p} y_1] \end{aligned}$$

Then αC is a p-convex set. ■

Corollary 2.3 If C_1 and C_2 are p-convex sets, then $\alpha C_1 + \beta C_2$ is also a p-convex set, where α, β are scalar. That is, the family of p-convex sets form a vector space.

Proof. It is obvious. ■

If X is a topological vector space and A is a subset of X , the “closed p-convex hull”, denoted by “p-cvx”, is the smallest closed p-convex set containing A .

Let $x_1, x_2, \dots, x_n \in A$, where A is p-convex. Given $t_j \geq 0$ such that $\sum_1^n t_j^p = 1$, then $\sum_1^n t_j x_j$ is said to be a “p-convex combination of x_j .”

III. P-CONVEX FUNCTIONS AND P-EPIGRAPH.

In this paper, we extend some concepts and theorems to non convex analysis. In fact, we have proved that the family of p-convex sets form a vector space, (Th.2.3).

The epigraph of f is defined to be the set of all points lying on or above its graph. We proved that, if S is a nonempty p-convex set in R^n and $f: S \rightarrow R^+$. Then f is p-convex if and only if the epigraph of f , p-epi f , is a p-convex set (Th. 3.3). As an example of a p-convex function is $f(x) = \|x\|_p$ defined by a p-norm on a vector space (Example 3.1).

We also proved equivalent properties to A-p-convex sets for some fixed set A of E , (Th.3.5). Here the set B is said to be A-p-convex set if $B = A \cap C$ for some p-convex set C in E .

Now for two fixed subsets A and Q of a vector space E if an operator between two power sets,

$$\gamma = \gamma_{PQ}: P(E) \rightarrow P(A),$$

is defined by

$$\gamma(B) = p - cvx(B \cap Q) \cap A$$

for a real vector space E and for any subset A of E we proved that B is a A-p-convex if and only if $B = \gamma(B)$ for all $Q \supseteq A$, (Th. 3.5)

Finally, we proved that the intersection $\cap B_j, j \in J$ of A-p-convex sets B_j is A-p-convex, (Th.3.8).

Example 3.1. The following is an example of a p-convex function. let

$$f: B_{lp2} \rightarrow R \text{ with } f(x) = \|x\|_p, x \in B_{lp2},$$

In fact,

$$\begin{aligned} f((1-t)^{1/p} x \\ + t^{1/p} y) &= \|(1-t)^{1/p} x + t^{1/p} y\|_p \\ &\leq \|(1-t)^{1/p} x\|_p + \|t^{1/p} y\|_p \end{aligned}$$

i.e. $f(x)$ is a p-convex function. ■

Example 3.2. The following is an example of $1/2$ -convex function

$$f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}, x_1, x_2 \geq 0$$

In fact,

$$f((1-t)^2 x + t^2 y) = f((1-t)^2 (x_1, x_2) + t^2 (y_1, y_2))$$

$$= f((1-t)^2 x_1, ((1-t)^2 x_2) + (t^2 y_1, t^2 y_2))$$

$$= f((1-t)^2 x_1 + t^2 y_1, ((1-t)^2 x_2 + t^2 y_2))$$

$$\begin{aligned} &= \sqrt{(1-t)^2 x_1 + t^2 y_1} + \\ &\sqrt{(1-t)^2 x_2 + t^2 y_2} \end{aligned}$$

$$\leq (1-t) \sqrt{x_1} + t \sqrt{y_1}$$

$$+ (1-t) \sqrt{x_2} + t \sqrt{y_2}$$

$$\leq (1-t)(\sqrt{x_1} + \sqrt{x_2}) + t(\sqrt{y_1} + \sqrt{y_2})$$

$$\leq (1-t)f(x) + t f(y). \quad \blacksquare$$

Let S be a non convex set of a vector space E and $f: S \rightarrow R$; “the epigraph of f ” denoted by p-epi f , is a subset of $E \times R$ defined by,

$$\text{p-epi } f = \{(x, t); x \in S, t \in R, f(x) \leq t\}$$

That is, the epigraph of f is the set of all points lying on or above its graph.

Theorem 3.3. Let S be a nonempty p-convex set in R^n and $f: S \rightarrow R^+$. Then f is p-convex if and only if p-epi f is a p-convex set.

Proof. Assume f is a p-convex function. Let $(a, b) \in S$ and $(a, \alpha), (a, \beta)$ in p-epi f . Then for any $t \in [0, 1]$, we have

$$f\left(\frac{(1-t)^{1/p}a + t^{1/p}b}{(1-t)\alpha + t\beta}\right) \leq (1-t)f(a) + tf(b)$$

Since S is a p -convex set, $(1-t)^{1/p}a + t^{1/p}b \in S$.
Therefore,

$$\left(\frac{(1-t)^{1/p}a + t^{1/p}b}{(1-t)\alpha + t\beta}\right) \in p\text{-epi}f,$$

i.e., $p\text{-epi}f$ is a p -convex set.

To show sufficiency, assume that $p\text{-epi}f$ is a p -convex set, and consider $a, b \in S$ and

$$(a, f(a)), (b, f(b)) \in p\text{-epi}f$$

From p -convexity of $p\text{-epi}f$, for $t \in [0, 1]$, we have

$$\left(\frac{(1-t)^{1/p}a + t^{1/p}b}{(1-t)\alpha + t\beta}, \frac{(1-t)\alpha f(a) + t\beta f(b)}{(1-t)\alpha + t\beta}\right) \in p\text{-epi}f.$$

Since,

$$f\left(\frac{(1-t)^{1/p}a + t^{1/p}b}{(1-t)\alpha + t\beta}\right) \leq \frac{(1-t)\alpha f(a) + t\beta f(b)}{(1-t)\alpha + t\beta},$$

it follows that f is a p -convex function. ■

Corollary 3.4. If f and g are p -convex functions, then $\alpha f + \beta g$ is also a p -convex function, when $\alpha, \beta \geq 0$.
Proof.

$$\begin{aligned} & (\alpha f + \beta g)\left(\frac{(1-t)^{1/p}x + t^{1/p}y}{(1-t)\alpha + t\beta}\right) \\ &= \frac{(\alpha f)\left(\frac{(1-t)^{1/p}x + t^{1/p}y}{(1-t)\alpha + t\beta}\right) + (\beta g)\left(\frac{(1-t)^{1/p}x + t^{1/p}y}{(1-t)\alpha + t\beta}\right)}{(1-t)\alpha + t\beta} \\ &\leq \frac{\alpha \left[\frac{(1-t)^{1/p}}{\alpha} f(x) + \frac{t^{1/p}}{\beta} f(y)\right] + \beta \left[\frac{(1-t)^{1/p}}{\alpha} f(x) + \frac{t^{1/p}}{\beta} f(y)\right]}{(1-t)\alpha + t\beta} \end{aligned}$$

since f and g are p -convex. Therefore $\alpha f + \beta g$ is also p -convex. ■

Let E be a real vector space and fix a subset A of E . A set B of E is said to be “**A-p-convex**” if there exist a p -convex set C in E such that

$$B = C \cap A.$$

We are interested in the case $E = \mathbb{R}^n$ and $A = \mathbb{Z}^n$.

For digitization of p -convex sets the mapping $C \rightarrow C \cap \mathbb{Z}^n$ is not always satisfactory, because it yields the empty set for some long and narrow p -convex sets C . One might then want to replace it by a mapping like

$$C \rightarrow (C + D) \cap \mathbb{Z}^n$$

where D is some fixed set which grants that the image is nonempty when C is nonempty.

Theorem 3.5. Given a vector space E and a subset A of E , the following properties are equivalent for any subset B of A .

1. B is A - p -convex.
2. $B = A \cap (p\text{-cvx} B)$.

$$3. B \supseteq A \cap (p\text{-cvx} B)$$

4. For all n , all $a_0, a_1, \dots, a_n \in A$, and for all nonnegative numbers t_0, t_1, \dots, t_n with $\sum_{j=0}^n t_j^p = 1$, If $\sum_{j=0}^n t_j a_j \in A$, then $\sum_{j=0}^n t_j a_j \in B$.

Proof. This is obvious. As far as property 4 is concerned, we can in view generalize of Caratheodory's theorem, let n be the dimension of E if the space is finite dimensional; otherwise we must use all n . ■

Let us now fix two subsets A and Q of a vector space E and define an operator between two power sets,

$$\gamma = \gamma_{PQ} : P(E) \rightarrow P(A),$$

by

$$\gamma(B) = p\text{-cvx}(B \cap Q) \cap A.$$

We may consider $E = \mathbb{R}^n$, $A = m\mathbb{Z}^n$, $m = 1, 2, \dots$ and $Q = \mathbb{Z}^n$. Note that $\gamma(C)$ is A - p -convex if C is p -convex in \mathbb{R}^n .

Theorem 3.6. Let E be a real vector space and A any subset of E . Then B is A - p -convex if and only if $B = \gamma(B)$ for all $Q \supseteq A$, also if and only if $B = \gamma(B)$ for some $Q \supseteq A$.

Proof. If B is A - p -convex, then $B = C \cap A$ for some p -convex C . Now,

$$\begin{aligned} \gamma(B) &= \gamma(C \cap A) = p\text{-cvx}(C \cap A \cap Q) \cap A \\ &= p\text{-cvx}(C \cap A) \cap A \end{aligned}$$

$$= p\text{-cvx}(B) \cap A = C \cap A = B, \quad \text{for all } Q \supseteq A.$$

If $B = \gamma(B)$ for some choice of $Q \supseteq A$, then $B = p\text{-cvx}(B \cap Q) \cap A = C \cap A$.

Defining $C = p\text{-cvx}(B \cap Q)$ so that B is A - p -convex. ■

Corollary 3.7. If $B = C \cap A$, then $Q \supseteq \gamma(B)$ for any Q .

Proof. Note that in the previous definition of A - p -convex sets we may take

$$C = \gamma(B) = p\text{-cvx} B$$

Provided that $Q \supseteq A$. ■

Theorem 3.8. Let E be a vector space and A any subset of E . If $B_j, j \in J$, are A - p -convex sets, then the intersection $\bigcap B_j$ is a A - p -convex set. If the index set J is ordered and filtering to right and if $(B_j)_j$ is an increasing family of A - p -convex sets, then the union $\bigcup B_j$ is also p -convex.

Proof. For each B_j , we have

$$B_j = C_j \cap A,$$

where $C_j = p\text{-cvx}$ is a p -convex set in E . Then

$$\bigcap B_j = \bigcap (C_j \cap A) = (\bigcap C_j) \cap A.$$

For the union we have

$$\cup B_j = \cup (C_j \cap A) = (\cup C_j) \cap A. \quad \blacksquare$$

IV. CONCLUSION

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions

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