Directional q-Derivative

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***Abstract*— In this paper partial q-derivative of a two variable function f and directional q-derivative of function f at the point $P = (p_1, p_2)$ in the direction of a unit vector are introduced and some properties of q- directional derivative are investigated.**

***Index Terms*—Partial q-derivative, directional q- derivative.**

I. INTRODUCTION

A quantum calculus is a version of calculus in which we do not take limits. Derivatives are differences and anti derivatives are sums. It is a theory, where smoothness is no more required[1].

The general idea in this paper is to generalize the concept d-derivative of a real function f to a two variable function and to construct q- directional derivative of a function.

II. PRELIMINARIES

Consider an arbitrary function $f(x)$. The q-derivative $D_q f$ of the function $f(x)$ is given by

$$\left(D_q f\right)(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad (1)$$

if $x \neq 0$ and $\left(D_q f\right)(0) = f'(0)$ provided $f'(0)$ exists. Note that

$$\lim_{q \to 1} D_q f(x) = \frac{d}{dx} f(x)$$

if $f(x)$ differentiable[2]. The Leibniz notation $\frac{d}{dx} f(x)$, a ratio of two "infinitesimals" is rather confusing, since the notion of the differential $df(x)$ requires an elaborate explanation. In contrast, the notion of q-differential is obvious and plain ratio[3].

It is clear that as with ordinary derivative, the action of taking the q-derivative of a function is a linear operator. In other words, for any constants a and b, we have

$D_q\{af(x) + bg(x)\} = aD_q f(x) + bD_q g(x)$

The formulas for the q-derivative of a product and a quotient of functions are [3]

$D_q\{f(x)g(x)\} = D_q f(x)g(x) + f(qx)D_q g(x)$

and

$$D_q\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}.$$

If f is q-differentiable at x, then

$f(qx) = f(x) + (q-1)xD_q f(x)$.

The q-analogue of the chain rule is more complicated since it involves q-derivatives for different values of q depending on the composed functions. The chain rule for general functions $f(x)$ and $g(x)$ is[3]

$D_q(f \circ g)(x) = D_{\frac{g(qx)}{g(x)}} f\big(g(x)\big) D_q\, g(x)$.

III. PARTIAL Q-DERIVATIVE

In this section we will define to partial q-derivative of a two variable functions by using the definition one variable case and give a version of a chain rule for two variable functions.

For i=1,2, I_i is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set

$I^2 = I_1 \times I_2 = \{t = (t_1, t_2) : t_i \in I_i, i = 1,2\}$.

Definition 3.1. Let $f: I^2 \to \mathbb{R}$ be a two variable function. The partial q-derivative of f with respect to t_1 and t_2 is defined by

$$\frac{\partial f(t)}{\partial_{q_1} t_1} = \frac{f(q_1 t_1, t_2) - f(t_1, t_2)}{q_1 t_1 - t_1}$$

and

$$\frac{\partial f(t)}{\partial_{q_2} t_2} = \frac{f(t_1, q_2 t_2) - f(t_1, t_2)}{q_2 t_2 - t_2}$$

respectively.

Note that

$$\lim_{q_i \to 1} \frac{\partial f(t)}{\partial_{q_i} t_i} = \frac{\partial f(t)}{\partial t_i}, \qquad i = 1,2$$

if $f(t)$ differentiable.

Lemma 3.2 Let $f, g: I^2 \to \mathbb{R}$ are two variable functions. Then, for $a, b \in \mathbb{R}$, i=1,2,

$$\frac{\partial}{\partial_{q_i} t_i}\{af(t) \pm bg(t)\} = a\frac{\partial f(t)}{\partial_{q_i} t_i} \pm b\frac{\partial g(t)}{\partial_{q_i} t_i}$$

and

$$\frac{\partial}{\partial_{q_1} t_1}\{f(t)g(t)\} = f(q_1 t_1, t_2)\frac{\partial g(t)}{\partial_{q_1} t_1} + g(t)\frac{\partial f(t)}{\partial_{q_1} t_1},$$

$$\frac{\partial}{\partial_{q_2} t_2}\{f(t)g(t)\} = f(t_1, q_2 t_2)\frac{\partial g(t)}{\partial_{q_2} t_2} + g(t)\frac{\partial f(t)}{\partial_{q_2} t_2}.$$

Proof: By the Definition 3.1. we get easily linearity. And the partial q-derivative of product f and g is

$$\frac{\partial (fg)(t_1, t_2)}{\partial_{q_1} t_1} = \frac{(fg)(q_1 t_1, t_2) - (fg)(t_1, t_2)}{q_1 t_1 - t_1}$$

$$= \frac{f(q_1 t_1, t_2)g(q_1 t_1, t_2) - f(t_1, t_2)g(t_1, t_2)}{q_1 t_1 - t_1}$$

$$= \frac{f(q_1 t_1, t_2)g(q_1 t_1, t_2) - f(t_1, t_2)g(t_1, t_2) \pm f(q_1 t_1, t_2)g(t_1, t_2)}{q_1 t_1 - t_1}$$

$$= \frac{f(q_1 t_1, t_2)g(q_1 t_1, t_2) - f(q_1 t_1, t_2)g(t_1, t_2)}{q_1 t_1 - t_1} + \frac{f(q_1 t_1, t_2)g(t_1, t_2) - f(t_1, t_2)g(t_1, t_2)}{q_1 t_1 - t_1}$$

$$= f(q_1 t_1, t_2)\frac{\{g(q_1 t_1, t_2) - g(t_1, t_2)\}}{q_1 t_1 - t_1} + \frac{\{f(q_1 t_1, t_2) - f(t_1, t_2)\}}{q_1 t_1 - t_1} g(t_1, t_2)$$

$$= f(q_1 t_1, t_2)\frac{\partial g(t_1, t_2)}{\partial_{q_1} t_1} + g(t_1, t_2)\frac{\partial f(t_1, t_2)}{\partial_{q_1} t_1}.$$

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Lemma 3.3. Let $u_1(t)$ and $u_2(t)$ are real functions and $f: I^2 \to \mathbb{R}$ be a two variable function. Then $f(u_1(t), u_2(t))$ is a real function of variable t and

$D_q f(u_1(t), u_2(t)) = \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t) + \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t).$

Proof: Let $g(t) = f(u_1(t), u_2(t))$. Then the q-derivative of $g(t)$, we have

$$D_q g(t) = \frac{g(qt) - g(t)}{qt - t}$$

$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t}$$

$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(t)) \pm f(u_1(t), u_2(qt))}{qt - t}$$

$$= \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(qt))}{qt - t} + \frac{f(u_1(t), u_2(qt)) - f(u_1(t), u_2(t))}{qt - t} = R_1 + R_2.$$

By the chain rule, we obtain

$$R_1 = \frac{f(u_1(qt), u_2(qt)) - f(u_1(t), u_2(qt))}{qt - t} \frac{(u_1(qt) - u_1(t))}{(u_1(qt) - u_1(t))}$$

$$= \frac{f\left(u_1(qt)\frac{u_1(t)}{u_1(t)}, u_2(qt)\right) - f(u_1(t), u_2(qt))}{\left(\frac{u_1(qt)}{u_1(t)} - 1\right) u_1(t)} \frac{(u_1(qt) - u_1(t))}{qt - t}$$

$$= \frac{f(q_1^* u_1(t), u_2(qt)) - f(u_1(t), u_2(qt))}{(q_1^* - 1) u_1(t)} \frac{(u_1(qt) - u_1(t))}{qt - t}$$

$$= \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t)$$

and

$$R_2 = \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t)$$

where $q_1^* = \frac{u_1(qt)}{u_1(t)}$ and $q_2^* = \frac{u_2(qt)}{u_2(t)}$. Hence

$D_q f(u_1(t), u_2(t)) = \frac{\partial f(u_1(t), u_2(qt))}{\partial_{q_1^*} u_1} D_q u_1(t) + \frac{\partial f(u_1(t), u_2(t))}{\partial_{q_2^*} u_1} D_q u_2(t).$

IV. Directional q-Derivative

Definition 4.1. Let $f: I^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of f function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is defined as the number

$$\frac{\partial f(P)}{\partial_q \vec{v}} = D_q f(P + \lambda v)\big|_{\lambda=0}.$$

Theorem 4.2. Let $f: I^2 \to \mathbb{R}$ be a two variable function. The directional q-derivative of f function at the point $P = (p_1, p_2)$ in the direction of the unit vector $\vec{v} = (v_1, v_2)$ is

$$\frac{\partial f(P)}{\partial_q \vec{v}} = \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$

Proof: $D_q u_i(\lambda)\big|_{\lambda=0} = v_i$, since $u_i(\lambda) = p_i + \lambda v_i$,i=1,2. Then by the Lemma 3.3 the theorem is proved.

Theorem 4.3. Let $a, b \in \mathbb{R}$, $f, g: I^2 \to \mathbb{R}$ are two variable function, $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ are unit vectors. Then

i) $\frac{\partial f(P)}{\partial_q \overrightarrow{v+w}} = \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial f(P)}{\partial_q \vec{w}}$

ii) $\frac{\partial (f+g)(P)}{\partial_q \vec{v}} = \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial g(P)}{\partial_q \vec{v}}$

Proof: i) By the Definition 4.1 and Theorem 4.2, we have

$$\frac{\partial f(P)}{\partial_q \overrightarrow{v+w}} = \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(v_1 + w_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(v_2 + w_2)$$

$$= \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(v_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(v_2) + \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1}(w_1) + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2}(w_2)$$

$$= \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial f(P)}{\partial_q \vec{w}}.$$

ii)

$$\frac{\partial (f+g)(P)}{\partial_q \vec{v}} = \frac{\partial (f+g)(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial (f+g)(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$

$$= \frac{\partial f(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial f(p_1, p_2)}{\partial_{q_2^*} u_2} v_2 + \frac{\partial (g)(q_1^* p_1, p_2)}{\partial_{q_1^*} u_1} v_1 + \frac{\partial (g)(p_1, p_2)}{\partial_{q_2^*} u_2} v_2$$

$$= \frac{\partial f(P)}{\partial_q \vec{v}} + \frac{\partial g(P)}{\partial_q \vec{v}}$$

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