Mean Excess over a Threshold for Number of Customers in Discrete-time $Geo^X/D/$ Queue

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***Abstract*— This paper considers a discrete-time $Geo^X/D/$ queueing system where customers arrive at a facility with a single server according to a batch geometric process with customer service times assumed to be one slot. This paper investigates the mean excess over a threshold for the number of customers in the queueing system.**

***Index Terms*— Queueing system, batch geometric process, number of customers, mean excess.**

I. INTRODUCTION

Consider a classical discrete-time single server $Geo^X/D/1$ queueing system, consisting of a stream of customers arriving according to a batch geometric process and a server that processes these customers with deterministic service time of 1 slot. The random variable N_k is defined as the number of customers in the queueing system at the k-th slot boundary. Let $\tau(T)$ be the slot boundary of first entry into the interval (T,∞), $T>0$, of the number of customers. The distribution of the excess $L(T)$ at the first-passage time over the threshold T for the number of customers is defined as

$$L(T) \equiv N_{\tau(T)} - T. \tag{1}$$

That is,

$$P(L(T)=l) \equiv P(N_{\tau(T)} - T = l). \tag{2}$$

Ghost and Resnick [1] has investigated some theoretical and practical aspects of the use of the mean excess plot, a widely used tool in the study of risk, insurance and extreme values. Mijatovic and Pistorius [2] has established the existence of the weak limit of undershoots and overshoots of the reflected process of a Levy process as a threshold level tends to infinity and has provided explicit formulas for their joint cumulative distribution functions. They have applied their results to analyze the behavior of the classical continuous-time M/G/1 queueing system at buffer-overflow, both in a stable and unstable case. In M/G/1 queueing model with Weibull service time distribution, Bae and Park [3] has proposed the approximation of the distribution of the excess for the workload over a threshold at the moment where the workload process exceeds the threshold

In this paper we investigate the mean excess $\mathrm{E}(L(T))$ over

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The threshold T for the number of customers in the discrete-time single server $Geo^X/D/1$ queueing system.

The paper is organized as follows. In Section II, we described the discrete-time queueing system. In Section III, we investigate the mean access over a threshold for the number of customers in the queueing system. Conclusion is provided in Section IV.

II. SYSTEM MODEL

In this section, we formulate the queueing model, which is based on the following assumptions:

1. We consider a discrete-time queueing system in which the time axis is divided into fixed-length contiguous intervals, referred to as slots.
2. Customers arrive according to a batch geometric process.
3. The numbers of arrivals during the consecutive slots are assumed to be independent and identically distributed random variables with distribution $\{a_k, k=0,1,\dots\}$.
4. The service of a customer can start only at a slot boundary.
5. The service time of customers is one slot.
6. The system has a buffer of infinite capacity.
7. Customers are served in FCFS order.

III. ANALYSIS

The distribution of the excess $L(T)$ at the first-passage time over the threshold T for the number of customers is defined as (1). In order to write the equations in shorthand notation, in this section we omit the threshold T in the notation $L(T)$ and use just L instead of $L(T)$.

Let L_l, $l=1,2,\dots$, be the mean excess at the first-passage time over the threshold T for the number of customers, given the condition that there are l customers in the buffer and the number of customers exceeds the threshold T in this busy period. Then, the following system of linear equations holds:

$$\begin{aligned}
L_T &= a_0 L_{T-1} + a_1 L_T + a_2 \times 1 + a_3 \times 2 + + a_4 \times 3 \cdots \\
L_{T-1} &= a_0 L_{T-2} + a_1 L_{T-1} + a_2 L_T + a_3 \times 1 + a_4 \times 2 + \cdots \\
L_{T-2} &= a_0 L_{T-3} + a_1 L_{T-2} + a_2 L_{T-1} + a_3 L_T + a_4 \times 1 + \cdots \\
&\vdots \\
L_2 &= a_0 L_1 + a_1 L_2 + a_2 L_3 + \cdots + a_{T-1} L_T + a_T \times 1 + \cdots \\
L_1 &= \frac{a_1}{1-a_0} L_1 + \cdots + \frac{a_T}{1-a_0} L_T + \frac{a_{T+1}}{1-a_0} \times 1 \cdots
\end{aligned}$$

The mean excess $\mathrm{E}(L(T))$ at the first-passage time over the threshold T for the number of customers is given by

$$\mathrm{E}\big(L(T)\big) = \sum_{l=1}^{T} \frac{a_l}{1-a_0} L_l + \sum_{l=T+1}^{\infty} l \frac{a_{T+l}}{1-a_0}. \tag{3}$$

Hence,

$$\mathrm{E}\big(L(T)\big) = L_1, \tag{4}$$

which is obtained recursively from the above system of linear equations.

Here are some examples. If

$$a_0 = \frac{1}{2}, a_2 = \frac{1}{2}, \tag{5}$$

then

$$\begin{aligned} L_T &= \frac{1}{2} L_{T-1} + \frac{1}{2} \\ L_{T-1} &= \frac{1}{2} L_{T-2} + \frac{1}{2} L_T \\ L_{T-2} &= \frac{1}{2} L_{T-3} + \frac{1}{2} L_{T-1} \\ &\vdots \\ L_2 &= \frac{1}{2} L_1 + \frac{1}{2} L_3 \\ L_1 &= L_2. \end{aligned}$$

Hence,

$$\mathrm{E}\big(L(T)\big) = 1. \tag{6}$$

If

$$a_0 = \frac{1}{2}, a_3 = \frac{1}{2}, \tag{7}$$

then

$$\begin{aligned} L_T &= \frac{1}{2} L_{T-1} + 1 \\ L_{T-1} &= \frac{1}{2} L_{T-2} + \frac{1}{2} \\ L_{T-2} &= \frac{1}{2} L_{T-3} + \frac{1}{2} L_T \\ &\vdots \\ L_2 &= \frac{1}{2} L_1 + \frac{1}{2} L_4 \\ L_1 &= L_3. \end{aligned}$$

After some manipulation, we get

$$L_l = b_l L_T - c_l, l = T, T-1, \dots, 1, \tag{8}$$

and

$$L_1 = L_3, \tag{9}$$

where the sequence $b_T, b_{T-1}, \dots, b_1$ is the first T terms of the sequence

$$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, \dots, \tag{10}$$

and the sequence $c_T, c_{T-1}, \dots, c_1$ is the first T terms of the sequence

$$0, 2, 5, 10, 18, 31, 52, 86, 141, 230, \dots. \tag{11}$$

Note that the increments of each sequence are a part of Fibonacci sequence, characterized by the fact that every term after the first two is the sum of the twopreceding ones, and the sequence (10) counts the number of Fibonaccu meanders. In the case of (7), when $T = 1$, trivially $\mathrm{E}\big(L(T)\big) = 2$; when $T = 2$, trivially $\mathrm{E}\big(L(T)\big) = 1$; and when

$$T = 3, \tag{12}$$

$$\mathrm{E}\big(L(T)\big) = \frac{5}{3}. \tag{13}$$

IV. Conclusion

This paper has considered the analysis of mean excess over a threshold for the number of discrete-time $Geo^X/D/1$ queueing system where customers arrive at a facility with a single server according to a batch geometric process with customer service times assumed to be one slot. This paper has derived the system of linear equations for the mean excess and also presented some examples.

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References

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