Optimal Investment and Optimal Reinsurance

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Abstract—In this paper, Aiming at the delay claim risk model, the optimal investment and optimal reinsurance strategy which makes the expected index utility maximum of the final wealth are studied under the principle of variance premium principle. Firstly, when the optimal reinsurance is in the form of proportional reinsurance, we use diffusion approximation to approximate the claim process of insurance companies, then, we use the dynamic programming principle to solve the Hamilton-Jacobi-Bellman equation. Finally, we obtain the explicit optimal strategy and the value function.

Index Terms—variance premium principle; reinsurance; diffusion approximation; HJB equation; index utility function

I. INTRODUCTION

In recent years, in the improvement of the classical model, the delay risk model is a kind of risk model which is closer to the actual insurance, it is also a situation often encountered by insurance companies in the claim process. After the occurrence, the guarantor not only has to pay for the loss of the car but also if he buys the third party insurance, the guarantor will pay for the third party after a period of time. For this kind of situation, Waters et al [1] proposed a risk model with delay claim. Many scholars at home and abroad have produced a strong interest in it. Yuen et al [2] using martingale method to study the final ruin probability of the delay risk model. Xie et al [3] discussed the expected discounting of total dividend for the delay risk model with stochastic interest rate. Recently, Xiao et al [4] studied the limit property of the delayed risk model with dependent compensation and investment.

In real life, the huge claim caused by the frequent occurrence of accidental catastrophe makes the insurance company lose serious damage, and reinsurance is an important means to prevent and resolve huge risks. Therefore, the research on reinsurance has gradually gained the attention of the insurance company, and the theoretical operation of the reinsurance is increasing. The key problem is the optimal reinsurance in the reinsurance, which is to consider the form of reinsurance and the amount of specific reinsurance, to this end, the insurer needs to balance the risk and return and make the most reasonable decision possible. we also know in the insurance practice, because of fierce competition, it is difficult to satisfy the insurance company's payment only by the premium income. Therefore, the insurance companies must invest the surplus and obtain a large amount of income from the investment to improve their solvency. Actually, the optimal reinsurance and optimal investment have been one of the main research contents of the actuarial study. Schmidli H [5] studies the optimal proportional reinsurance problem. Hipp and Taksar [6] study the problem of non proportional reinsurance. Cao et al [7] study the optimal reinsurance and optimal investment problem with the minimum ruin probability. Zhang et al [8] studies the problem of reinsurance and finite time ruin probability. Yang and Zhang [9] assume that the price of risk assets satisfies the jump diffusion model, consider the optimal investment problem of the ultimate wealth utility maximization. Lin and Li [10] assume that insurance companies can invest in reinsurance business on the basis of Yang and Zhang [9] model's, and study the optimal investment and optimal reinsurance strategy of insurance companies under the risk asset price satisfaction jump diffusion model. Centeno [11] studies dependent risks and excess of loss reinsurance. Bai et al [12] study the optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. Liang and Yuen [13] respectively study the optimal proportional reinsurance of the jump model and the diffusion approximation model under the optimization criterion of the variance premium principle and the maximum final wealth expectation index utility. Yuen et al [14] adopt the same optimization criterion, but it pushes the dependent double insurance model to a more practical dependent multiple insurance model, respectively study the optimal results of the jump model and the diffusion approximation model under the expected value premium principle. Recently, Zhang and Xiao [15] study the optimal reinsurance of the dependent multiple insurance model under the principle of variance premium.

This paper applies the idea of [15] to the delay claim risk model, and studies the optimal investment and optimal reinsurance strategy which makes the expected index utility maximum of the final wealth under the variance premium principle when the number of claim is homogeneous Poisson process. Through diffusion approximation and use the dynamic programming principle, and we obtain an explicit expression of the optimal investment strategy and the optimal reinsurance strategy and the maximum index expected utility function. Finally, we discuss the effect of the parameters on the optimal strategy is illustrated by simulation. The rest of the structure of this article is arranged as follows. In Section 2 the basic structure of the model is given. In Section 3, we give the optimal strategies and value function.

II. MODEL FORMULATION AND DIFFUSION APPROXIMATION

This paper assumes that all random processes and random variables are defined in a complete probability space \((\Omega, \mathcal{F}, P)\) and there is a \(\sigma\)-filter \(\{F_t, t \geq 0\}\) that satisfies the usual condition, that is \(F_t\) right continuous and \(P\) complete. This article also allows continuity transactions, excluding transaction costs and taxes, and all assets are infinitely separable.

Consider the following delay risk model, \(U(t)\) represents the surplus process of insurance companies at \(t\):
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\[ U(t) = u + ct - \sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t)} Y_i. \]  

(1)

Where

(i) \( u \) is the amount of initial surplus and \( c \) is the rate of premium.

(ii) \( \{X_i; i = 1,2, \ldots \} \) is the main claim amount, and \( \{S_i; i = 1,2, \ldots \} \) is the time for the occurrence of the \( i \) main claim.

(iii) \( \{Y_i; i = 1,2, \ldots \} \) is the delay claim amount.

We assume the number processes are correlated in the way that

\[ N_2(t) = \sum_{i=1}^{N(t)} I_{\{S_i \leq t \}}. \]

So the surplus process \( U(t) \) can be expressed as

\[ U(t) = u + ct - \sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t)} Y_i I_{\{S_i \leq t \}}. \]

For the study of this paper, we make the following basic assumptions.

(A1) Let the main claim sequence \( \{X_i; i = 1,2, \ldots \} \) are independently and identically distributed in \( X \) , and with common distribution \( F \) . The delay claim sequence \( \{Y_i; i = 1,2, \ldots \} \) are independently and identically distributed in \( Y \) , and with common distribution \( G \) :Their first and second moments exist, and write and \( \mu_2(X), \mu_2(Y) \).

(A2) Delay payment interval \( \{T_i; i = 1,2, \ldots \} \) are independently and identically distributed in \( T \) , and with common distribution \( H \).

(A3) Claim amount \( X \) and \( Y \) are independent of each other and the counting process \( N(t) \) is a homogeneous Poisson process and

\[ E N_1(t) = \lambda t, \]

\[ E N_2(t) = E \left( \sum_{i=1}^{N(t)} I_{\{S_i \leq t \}} \right) = \lambda t p(t), \]

\[ p(t) = p[U + T \leq t] \] and \( U \) is a uniform distribution on \((0,1)\).

In order to guarantee huge losses, the insurance company arranged the reinsurance strategy of \( f_i \) at the \( t \) time risk level of self retention. For \( f_i \), we assume \( 0 \leq f_i \leq x \) and increase monotonously on \((0, \infty)\). we assume the reinsurance premium is calculated according to the principle of variance premium and the safety load is \( \theta > 0 \). we have

\[ E \left( \sum_{i=1}^{N(t)} (X_i - f_i(X_i)) \right) = \lambda t \mu_2^{(1)}(X), \]

\[ E \left( \sum_{i=1}^{N(t)} (Y_i - f_i(Y_i)) \right) = \lambda t p(t) \mu_2^{(1)}(Y). \]

\[ Var \left( \sum_{i=1}^{N(t)} (X_i - f_i(X_i)) \right) = \lambda t \mu_2^{(2)}(X), \]

\[ Var \left( \sum_{i=1}^{N(t)} (Y_i - f_i(Y_i)) \right) = \lambda t p(t) \mu_2^{(2)}(Y). \]

where \( M = X - f_i(X), N = Y - f_i(Y) \).

Assuming that \( \pi \) is a feasible strategy, the net premium rate of the original insurance company at time \( t \) is

\[ C_{\pi}(t) = c - \lambda \mu_2^{(1)} + \lambda p(t) \mu_2^{(1)} - \theta(\lambda \mu_2^{(1)}(X) + \lambda p(t) \mu_2^{(1)}(Y)) + 2 \lambda p(t) \mu_2^{(2)}(Y). \]

Therefore, the surplus process of the original insurance company is

\[ U_{\pi}(t) = u + \int_0^t C_{\pi}(s) ds - \left( \sum_{i=1}^{N(t)} f_i(X_i) \right) + \left( \sum_{i=1}^{N(t)} f_i(Y_i) \right). \]

For the delay claim risk model, it is difficult to solve the explicit solution, to make the problem can be processed and obtained explicit solution. we adopt the form of diffusion approximation. From Grandell, if \( \mathcal{N}(t) \) obey Poisson distribution of \( \lambda \) , we know that the Brownian motion claim process \( S(t) \) given by

\[ S(t) = at - \sigma B_t. \]

where \( a = \lambda E(X) \) and \( \sigma^2 = \lambda E(X^2) \) can be see as a diffusion approximation to the compound Poisson process. \( B_t \) is a standard Brownian motion.

So the surplus process \( U_{\pi}(t) \) diffusion approximation can be expressed as

\[ \tilde{U}_{\pi}(t) = u + \int_0^t C_{\pi}(s) ds - \left( \sum_{i=1}^{N(t)} f_i(X_i) \right) + \left( \sum_{i=1}^{N(t)} f_i(Y_i) \right) \]

\[ + \sqrt{\lambda \mu_2^{(2)} + \lambda p(t) \mu_2^{(2)}} + 2 \lambda p(t) \mu_2^{(2)} B_t. \]

Where

\[ \mathcal{E} \left( \sum_{i=1}^{N(t)} (f_i(X_i)) \right) = \lambda t \mu_2^{(1)}(X), \]

\[ Var \left( \sum_{i=1}^{N(t)} (f_i(X_i)) \right) = \lambda t p(t) \mu_2^{(2)}(X). \]

\[ B_t \] is a standard Brownian motion.

III Optimal strategies

Diffusion approximation provides a basis for studying optimal decision problems and obtaining explicit solutions. We assume that the insurer is coping with huge losses. they arrange a reinsurance business with a risk ratio of self retention \( q(t), 0 \leq q(t) \leq 1 \). we order \( f_i(\Lambda) = q(t) \) . we adopt the principle of variance premium principle, from Hipp and Taskar, we know the optimal reinsurance at this time is proportional reinsurance. Therefore, under the reinsurance strategy \( q(t) \), the surplus process \( \tilde{U}_{\pi}(t) \) satisfies the stochastic differential equation as follows
Suppose the insurance company has an exponential utility function at time $t$, if it follows investment strategy and reinsurance strategy of \( t \). The company is allowed to invest in the risky asset and free asset, is assumed to be a positive constant. The price of the risky asset is assumed to follow the stochastic differential equation

\[ dP(t) = rP(t)dt + \sigma_P(t)dW(t), \]

where \( r \) and \( \sigma_P(t) \) are positive constant represent the instantaneous rate of return of the risky asset and the volatility of the risky asset price respectively.

The goal of the insurance company is to maximize the expected utility at time \( t \), which is the only function of a fair premium independent of the earnings level of the insurance company under the zero utility principle, so it has an important position in the insurance mathematics and actuarial applications. See [15].

It can be obtained from the classical dynamic programming principle, see [16]. If the value function is continuous differentiable, then it will satisfy the following HJB equation.

\[ \sup_{b,q} A^{b,q}V(t,x) = 0. \]

and the boundary condition

\[ V(T,x) = u(x). \]

where

\[ A^{b,q}V(t,x) = V_t + (b(t)x_t + x_T(1-b(t))) + c - \lambda \mu^{(i)}(x_t) - \lambda p(t)u^{(i)}(1-q(t))^2) \]

\[ + \frac{1}{2}(q(t)^2 C + b^2 \sigma^2_t x^2) \] \( \forall x \), \( x > 0 \), \( u''(x) < 0 \) and \( u \) is an increasing concave function. The goal of the insurance company is to make a decision at time \( t \) to maximize the expected utility at time \( T \). We assume the surplus at \( t \) is \( x \), so the value function is

\[ V(t,x) = \sup_{b,q} E[u(X^{b,q}_T) \mid X^{b,q}_t = x]. \]

where \( m > 0 \), \( v > 0 \). The utility function has a constant absolute aversion coefficient \( v \), because the exponential utility is the only function of a fair premium independent of the insurance mathematics and actuarial applications. See [15].

In order to solve the optimization problem, we use dynamic programming method to get the following HJB equation, see [17].

\[ \sup_{b,q} \{ V_t + (b(t)x_t + x_T)(1-b(t))) + c - \lambda \mu^{(i)}(x_t) - \lambda p(t)u^{(i)}(1-q(t))^2) \]

\[ + \frac{1}{2}(q(t)^2 C + b^2 \sigma^2_t x^2) \] \( \forall x \), \( x > 0 \), \( u''(x) < 0 \) and \( u \) is an increasing concave function. The goal of the insurance company is to make a decision at time \( t \) to maximize the expected utility at time \( T \). We assume the surplus at \( t \) is \( x \), so the value function is

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\[-\lambda \mu^{(2)}_x - \lambda p(t) \mu^{(1)}_x - \theta(1 - q(t))^2 C (-ve^{(T-t)}) + \frac{1}{2} (q(t)^2 C + b^2 \sigma_1^2 x^2) (v^2 e^{2r(T-t)}) \]

Below we need to find \( b^*(t) \), \( q^*(t) \) that makes \( F(b(t), q(t)) \) take minimum value, for this, we find partial derivative of \( F(b(t), q(t)) \) on \( b(t) \) and \( q(t) \) respectively, and make it equal to 0, we get

\[
\left\{ \begin{array}{l}
\frac{\partial F}{\partial b} = (x_r - x) [-ve^{(T-t)}] + b \sigma_1^2 x^2 \left[ v^2 e^{2r(T-t)} \right] = 0, \\
\frac{\partial F}{\partial q} = -2\theta C (1 - q) \left[ v^2 e^{r(T-t)} \right] + q C \left[ v^2 e^{2r(T-t)} \right] = 0.
\end{array} \right.
\]

(7)

where \( C = \lambda (\mu^{(2)}_x + p(t) \mu^{(2)}_x + 2 p(t) \mu^{(1)}_x \mu^{(1)}_y) \).

By the first equation of (7), we have

\[ b^*(t) = \frac{r_1 - r}{x \nu \sigma_1^2 e^{r(T-t)}}. \]

By the second equation of (7), we have

\[ q^*(t) = \frac{2\theta}{2\theta + v e^{r(T-t)}}. \]

Putting \( q^*(t) \), \( b^*(t) \) into (6) yields

\[ h(T-t) = -\frac{(r_1 - r)^2}{2\sigma_1^2} (T-t) - \frac{e - \lambda \mu^{(1)}_x - \lambda p(t) \mu^{(1)}_x}{r} \nu (e^{r(T-t)} - 1) + \int_0^{T-t} k(s) ds. \]

With

\[ k(s) = -\theta (1 - q^*(t))^2 C (ve^{r_s}) + \frac{1}{2} (q^*(t)^2 C + b^*(t)^2 \sigma_1^2 x^2) (v^2 e^{2r_s}), \]

\[ C = \lambda (\mu^{(2)}_x + p(t) \mu^{(2)}_x + 2 p(t) \mu^{(1)}_x \mu^{(1)}_y). \]

Finally, we summarize the result of this section in the following theorem.

**Theorem 1** For any \( t \in [0, T] \), the optimal reinsurance strategy for the risk model is

\[ q^*(t) = \frac{2\theta}{2\theta + v e^{r(T-t)}}. \]

the optimal investment strategy is

\[ b^*(t) = \frac{r_1 - r}{x \nu \sigma_1^2 e^{r(T-t)}}. \]

and the value function is

\[ v(t, x) = \lambda_0 \frac{m}{v} \exp \{-v x e^{r(T-t)} + h(T-t) \}. \]

Where \( h(T-t) \) is defined in (8).

**Remark:** The optimal strategies for the diffusion model depends only on the safety loading \( \theta \), time \( T - t \), the expected instantaneous rate of return of the risky asset \( r_1 \), the volatility of the risky asset price \( \sigma_1 \) and interest rate \( r \). That is, the claim size distribution as well as the counting process have no effect on the optimal strategies.

**REFERENCES**


