Optimal Investment and Optimal Reinsurance

Yuedi Liu

Abstract— **In this paper, Aiming at the delay claim risk model, the optimal investment and optimal reinsurance strategy which makes the expected index utility maximum of the final wealth are studied under the principle of variance premium principle. Firstly, when the optimal reinsurance is in the form of proportional reinsurance, we use diffusion approximation to approximate the claim process of insurance companies. then, we use the dynamic programming principle to solve the Hamilton-Jacobi-Bellman equation. Finally, we obtain the explicit optimal strategy and the value function.**

Index Terms— **variance premium principle; reinsurance; diffusion approximation; HJB equation; index utility function**

I. INTRODUCTION

In recent years, in the improvement of the classical model, the delay risk model is a kind of risk model which is closer to the actual insurance, it is also a situation often encountered by insurance companies in the claim process. After the occurrence,~the guarantor not only has to pay for the loss of the car but also if he buys the third party insurance, the guarantor will pay for the third party after a period of time. For this kind of situation, Waters et al [1] proposed a risk model with delay claim. Many scholars at home and abroad have produced a strong interest in it. Yuen et al [2] using martingale method to study the final ruin probability of the delay risk model. Xie et al [3] discussed the expected discounting of total dividend for the delay risk model with stochastic interest rate. Recently, Xiao et al [4] studied the limit property of the delayed risk model with dependent compensation and investment.

In real life, the huge claim caused by the frequent occurrence of accidental catastrophe makes the insurance company lose serious damage, and reinsurance is an important means to prevent and resolve huge risks. Therefore, the research on reinsurance has gradually gained the attention of the insurance company, and the theoretical operation of the reinsurance is increasing. The key problem is the optimal reinsurance in the reinsurance, which is to consider the form of reinsurance and the amount of specific reinsurance, to this end, the insurer needs to balance the risk and return and make the most reasonable decision possible. we also know in the insurance practice, because of fierce competition, it is difficult to satisfy the insurance company's payment only by the premium income. Therefore, the insurance companies must invest the surplus and obtain a large amount of income from the investment to improve their solvency. Actually, the optimal reinsurance and optimal investment have been one of the main research contents of the actuarial study. Schmidli H [5] studies the optimal proportional reinsurance problem. Hipp and Taksar [6] study the problem of non proportional reinsurance. Cao et al [7] study the optimal reinsurance and optimal investment problem with the minimum ruin probability. Zhang et al [8] studies the problem of reinsurance and finite time ruin probability. Yang and Zhang [9] assume that the price of risk assets satisfies the jump diffusion model, consider the optimal investment problem of the ultimate wealth utility maximization. Lin and Li [10] assume that insurance companies can invest in reinsurance business on the basis of Yang and Zhang [9] model's, and study the optimal investment and optimal reinsurance strategy of insurance companies under the risk asset price satisfaction jump diffusion model. Centeno [11] studies dependent risks and excess of loss reinsurance. Bai et al [12] study the optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. Liang and Yuen [13] respectively study the optimal proportional reinsurance of the jump model and the diffusion approximation model under the optimization criterion of the variance premium principle and the maximum final wealth expectation index utility. Yuen et al [14] adopt the same optimization criterion, but it pushes the dependent double insurance model to a more practical dependent multiple insurance model,respectively study the optimal results of the jump model and the diffusion approximation model under the expected value premium principle. Recently, Zhang and Xiao [15] study the optimal reinsurance of the dependent multiple insurance model under the principle of variance premium.

This paper applies the idea of [15] to the delay claim risk model, and studies the optimal investment and optimal reinsurance strategy which makes the expected index utility maximum of the final wealth under the variance premium principle when the number of claim is homogeneous Poisson process. Through diffusion approximation and use the dynamic programming principle, and we obtain an explicit expression of the optimal investment strategy and the optimal reinsurance strategy and the maximum index expected utility function. Finally, we discuss the effect of the parameters on the optimal strategy is illustrated by simulation. The rest of the structure of this article is arranged as follows. In Section 2 the basic structure of the model is given. In Section 3, we give the optimal strategies and value function.

II. MODEL FORMULATION AND DIFFUSION APPROXIMATION

This paper assumes that all random processes and random variables are defined in a complete probability space (Ω, F, P) ,and there is a σ-filter $\{F_t, t \geq 0\}$ that satisfies the usual condition, that is F_t right continuous and P complete. This article also allows continuity transactions, excluding transaction costs and taxes, and all assets are infinitely separable.

Consider the following delay risk model, $U(t)$ represents the surplus process of insurance companies at t:

$$U(t)=u+ct-\sum_{i=1}^{N_1(t)}X_i-\sum_{i=1}^{N_2(t)}Y_i. \qquad (1)$$

Where

(i) u is the amount of initial surplus and c is the rate of premium.

(ii) $\{X_i; i=1,2\Lambda\}$ is the main claim amount, and $\{S_i; i=1,2\Lambda\}$ is the time for the occurrence of the i main claim.

(iii) $\{Y_i; i=1,2\Lambda\}$ is the delay claim amount.

We assume the number processes are correlated in the way that

$$N_2(t)=\sum_{i=1}^{N_1(t)}I_{\{S_i+T_i\le t\}}.$$

So the surplus process U(t) can be expressed as

$$U(t)=u+ct-\sum_{i=1}^{N_1(t)}X_i-\sum_{i=1}^{N_1(t)}Y_iI_{\{S_i+T_i\le t\}}.$$

For the study of this paper, we make the following basic assumptions.

(A1) Let the main claim sequence $\{X_i; i=1,2\Lambda\}$ are independently and identically distributed in X, and with common distribution F. The delay claim sequence $\{Y_i; i=1,2\Lambda\}$ are independently and identically distributed in Y, and with common distribution G .Their first and second moments exist, and write and $\mu_X^{(2)}$, $\mu_Y^{(2)}$.

(A2) Delay payment interval $\{T_i; i=1,2\Lambda\}$ are independently and identically distributed in T, and with common distribution H.

(A3) Claim amount X and Y are independent of each other and the counting process $N_1(t)$ is a homogeneous Poisson process and

$$EN_1(t)=\lambda t,$$

$$EN_2(t)=E(\sum_{i=1}^{N_1(t)}I_{\{S_i+T_i\le t\}})=\lambda tp(t),$$

$p(t)=p\{U+T\le t\}$ and U is a uniform distribution on (0,1).

In order to guarantee huge losses, the insurance company arranged the reinsurance strategy of f_t at the t time risk level of self retention. For f_t, we assume $0\le f_t\le x$ and increase monotonously on $(0,\infty)$. we assume the reinsurance premium is calculated according to the principle of variance premium and the safety load is $\theta>0$. we have

$$E(\sum_{i=1}^{N_1(t)}(X_i-f_t(X_i)))=\lambda t\mu_M^{(1)},$$

$$E(\sum_{i=1}^{N_2(t)}(Y_i-f_t(Y_i)))=\lambda tp(t)\mu_N^{(1)},$$

$$Var(\sum_{i=1}^{N_1(t)}(X_i-f_t(X_i)))=\lambda t\mu_M^{(2)}$$

$$Var(\sum_{i=1}^{N_2(t)}(Y_i-f_t(Y_i)))=\lambda tp(t)\mu_N^{(2)}.$$

where $M=X-f_t(X), N=Y-f_t(Y)$.

Assuming that π is a feasible strategy, the net premium rate of the original insurance company at time t is

$$C_\pi(t)=c-\lambda\mu_M^{(1)}-\lambda p(t)\mu_N^{(1)}-\theta(\lambda\mu_M^{(2)}+\lambda p(t)\mu_M^{(2)}+2\lambda p(t)\mu_M^{(1)}\mu_N^{(1)}).$$

Therefore, the surplus process of the original insurance company is

$$U_\pi(t)=u+\int_0^t C_\pi(s)ds-(\sum_{i=1}^{N_1(t)}f_t(X_i)+\sum_{i=1}^{N_2(t)}f_t(Y_i)).$$

For the delay claim risk model, it is difficult to solve the explicit solution, to make the problem can be processed and obtained explicit solution. we adopt the form of diffusion approximation. From Grandell, if $N(t)$ obey Poisson distribution of λ, we know that the Brownian motion claim process $S(t)$ given by

$$S(t)=at-\sigma B_t,$$

where $a=\lambda E(X)$ and $\sigma^2=\lambda E(X^2)$ can be see as a diffusion approximation to the compound Poission process. B_t is a standard Brownian motion.

So the surplus process $U_\pi(t)$ diffusion approximation can be expressed as

$$\tilde{U}_\pi(t)=u+\int_0^t C_\pi(s)ds-(\lambda\mu_{fX}^{(1)}+\lambda p(t)\mu_{fY}^{(1)})t+\sqrt{\lambda\mu_{fX}^{(2)}+\lambda p(t)\mu_{fY}^{(2)}+2\lambda p(t)\mu_{fX}^{(1)}\mu_{fY}^{(1)}}B_t.$$

Where

$$E(\sum_{i=1}^{N_1(t)}(f_t(X_i))=\lambda t\mu_{fX}^{(1)}, E(\sum_{i=1}^{N_2(t)}(f_t(Y_i))=\lambda tp(t)\mu_{fY}^{(1)}$$

$$Var(\sum_{i=1}^{N_1(t)}(f_t(X_i))=\lambda t\mu_{fX}^{(2)},$$

$$Var(\sum_{i=1}^{N_2(t)}(f_t(Y_i))=\lambda tp(t)\mu_{fY}^{(2)}.$$

B_t is a standard Brownian motion.

III Optimal strategies

Diffusion approximation provides a basis for studying optimal decision problems and obtaining explicit solutions. We assume that the insurer is coping with huge losses. they arrange a reinsurance business with a risk ratio of self retention $q(t)$, $0\le q(t)\le 1$, we order $f_t(x)=q(t)x$. we adopt the principle of variance premium principle, from Hipp and Taskar, we know the optimal reinsurance at this time is proportional reinsurance. Therefore, under the reinsurance strategy $q(t)$, the surplus process $\tilde{U}_\pi(t)$ satisfies the stochastic differential equation as follows

$$d\tilde{U}_{\pi}(t) = (c - \lambda\mu_X^{(1)} - \lambda p(t)\mu_Y^{(1)} - \theta\lambda(1-q(t))^2 (\mu_X^{(2)} + p(t)\mu_Y^{(2)} + 2p(t)\mu_X^{(1)}\mu_Y^{(1)}))dt + \sqrt{\lambda p(t)^2(\mu_X^{(2)} + p(t)\mu_Y^{(2)} + 2p(t)\mu_X^{(1)}\mu_Y^{(1)})}dB_t$$

with initial condition $\tilde{U}_{\pi}(0) = u$.

For simplicity, we assume that there are only two assets available for investors in the financial market: a risk-free asset, whose price at time t is denoted by $B(t)$ and risky asset, whose price at time t is denoted by $P(t)$. $B(t)$ is assumed to follow

$$dB(t) = rB(t)dt.$$

where r the risk-free interest rate, is assumed to be a positive constant. The price of the risky asset is assumed to follow the stochastic differential equation

$$dP(t) = r_1 P(t)dt + \sigma_1 P(t)dW_t.$$

where r_1 and σ_1 are positive constant represent the expected instantaneous rate of return of the risky asset and the volatility of the risky asset price respectively.

The company is allowed to invest in the risky asset and risk-free asset, and let $b(t)$ indicates the proportion of insurance companies' surplus invest in risky market, the rest of surplus invest in risk-free market. Let $X_t^{b,q}$ denote the wealth of the company at time t, if it follows investment policy $b(t)$, with initial wealth $X_0^{b,q} = x$, This process then evolves as

$$dX_t^{b,q} = b(t)X_t^{b,q}\frac{dP(t)}{P(t)} + (1-b(t))X_t^{b,q}\frac{dB(t)}{B(t)} + d\tilde{U}_{\pi}(t)$$

or, more explicitly,

$$dX_t^{b,q} = (b(t)X_t^{b,q}r_1 + X_t^{b,q}r(1-b(t)), + c - \lambda\mu_X^{(1)} - \lambda p(t)\mu_Y^{(1)} - \theta(1-q(t))^2 C)dt + b(t)X_t^{b,q}\sigma_1 dW_t + \sqrt{q(t)^2 C}dB_t$$

$$X_0^{b,q} = x,$$

where $C = \lambda(\mu_X^{(2)} + p(t)\mu_Y^{(2)} + 2p(t)\mu_X^{(1)}\mu_Y^{(1)})$.

This paper assumes that the investment strategy and reinsurance strategy of $b(t)$, $q(t)$ are feasible. and assuming that the insurer wants to maximize the final value of the surplus, that to say, surplus value of the expected utility at T time is $u(x)$, and $u'(x) > 0, u''(x) < 0$ and u is an increasing concave function. The goal of the insurance company is to make a decision at time t to make the maximum expected utility at T time. we assume the surplus at t time is x, so the value function is

$$V(t,x) = \sup_{b,q} E[u(X_T^{b,q}) \mid X_t^{b,q} = x].$$

Suppose the insurance company has an exponential utility function

$$u(x) = \lambda_0 - \frac{m}{v}e^{-vx}.$$

where $m > 0$, $v > 0$, The utility function has a constant absolute aversion coefficient v, because the exponential utility is the only function of a fair premium independent of the earnings level of the insurance company under the zero utility principle, so it has an important position in the insurance mathematics and actuarial applications. see [15].

It can be obtained from the classical dynamic programming principle, see [16]. If the value function is two continuous differentiable, then it will satisfy the following HJB equation.

$$\sup_{b,q} = A^{b,q}V(t,x) = 0. \tag{2}$$

and the boundary condition

$$V(T,x) = u(x).$$

Where

$$A^{b,q}V(t,x) = V_t + (b(t)xr_1 + xr(1-b(t)) + c - \lambda\mu_X^{(1)} - \lambda p(t)\mu_Y^{(1)} - \theta(1-q(t))^2 C)V_x + \frac{1}{2}(q(t)^2 C + b^2\sigma_1^2 x^2)V_{xx},$$

V_t, V_x, V_{xx} is expressed respectively the first derivative of the value function about t, the first derivative of x, the two derivative of x.

In order to solve the optimization problem, we use dynamic programming method to get the following HJB equation, see [17].

$$\sup_{b,q}\{V_t + (b(t)xr_1 + xr(1-b(t)) + c - \lambda\mu_X^{(1)} - \lambda p(t)\mu_Y^{(1)} - \theta(1-q(t))^2 C)V_x + \frac{1}{2}(q(t)^2 C + b^2\sigma_1^2 x^2)V_{xx}\} = 0 \tag{3}$$

and the boundary condition

$$V(T,x) = u(x).$$

In order to obtain the solution of the above equation, it is inspired by Browne see [18], this paper conjecture that its solution satisfies

$$V(t,x) = \lambda_0 - \frac{m}{v}e^{-vxe^{r(T-t)+h(T-t)}}, \tag{4}$$

Where $h(\cdot)$ is a deterministic function, which makes (4) the solution of equation (2), and obtains $h(0) = 0$ from the boundary condition $V(T,x) = u(x)$.

It can be obtained by (4), we have

$$\begin{aligned} V_t &= [V(t,x) - \lambda_0][vxre^{r(T-t)} - h'(T-t)], \\ V_x &= [V(t,x) - \lambda_0][-ve^{r(T-t)}], \\ V_{xx} &= [V(t,x) - \lambda_0][v^2 e^{2r(T-t)}]. \end{aligned} \tag{5}$$

we replace (5) into the equation (3) will be obtain

$$\inf_{b,q}\{-h'(T-t) + (b(t)xr_1 - xrb(t) + c - \lambda\mu_X^{(1)} - \lambda p(t)\mu_Y^{(1)} - \theta(1-q(t))^2 C)(-ve^{r(T-t)}) + \frac{1}{2}(q(t)^2 C + b^2\sigma_1^2 x^2)(v^2 e^{2r(T-t)})\} = 0. \tag{6}$$

We order

$$F(b(t), q(t)) = -h'(T-t) + (b(t)xr_1 - xrb(t) + c$$

$$-\lambda\mu_X^{(1)}-\lambda p(t)\mu_Y^{(1)}-\theta(1-q(t))^2C)(-ve^{r(T-t)})$$
$$+\frac{1}{2}(q(t)^2C+b^2\sigma_1^2x^2)(v^2e^{2r(T-t)}).$$

Below we need to find $b^*(t)$, $q^*(t)$ that makes $F(b(t),q(t))$ take minimum value, for this, we find partial derivative of $F(b(t),q(t))$ on b(t) and q(t) respectively, and make it equal to 0, we get

$$\begin{cases}\dfrac{\partial F}{\partial b}=(xr_1-xr)[-ve^{r(T-t)}]+b\sigma_1^2x^2[v^2e^{2r(T-t)}]=0,\\ \dfrac{\partial F}{\partial q}=-2\theta C(1-q)[ve^{r(T-t)}]+qC[v^2e^{2r(T-t)}]=0.\end{cases} \quad (7)$$

where $C=\lambda(\mu_X^{(2)}+p(t)\mu_Y^{(2)}+2p(t)\mu_X^{(1)}\mu_Y^{(1)})$.

By the first equation of (7), we have

$$b^*(t)=\frac{r_1-r}{xv\sigma_1^2e^{r(T-t)}},$$

By the second equation of (7), we have

$$q^*(t)=\frac{2\theta}{2\theta+ve^{r(T-t)}}.$$

Putting $q^*(t)$, $b^*(t)$ into(6)yields

$$h(T-t)=-\frac{(r_1-r)^2}{2\sigma_1^2}(T-t)$$
$$-\frac{c-\lambda\mu_X^{(1)}-\lambda p(t)\mu_Y^{(1)}}{r}v(e^{r(T-t)}-1)$$
$$+\int_0^{T-t}k(s)ds\ , \quad (8)$$

With

$$k(s)=-\theta(1-q^*(t))^2C(ve^{rs})+\frac{1}{2}(q^*(t)^2C$$
$$+b^*(t)^2\sigma_1^2x^2)(v^2e^{2rs}),$$
$$C=\lambda(\mu_X^{(2)}+p(t)\mu_Y^{(2)}+2p(t)\mu_X^{(1)}\mu_Y^{(1)}).$$

Finally, we summarize the result of this section in the following theorem.

Theorm 1 For any $t\in[0,T]$, the optimal reinsurance strategy for the risk model is

$$q^*(t)=\frac{2\theta}{2\theta+ve^{r(T-t)}},$$

the optimal investment strategy is

$$b^*(t)=\frac{r_1-r}{xv\sigma_1^2e^{r(T-t)}},$$

and the value function is

$$v(t,x)=\lambda_0-\frac{m}{v}\exp\{-vxe^{r(T-t)}+h(T-t)\}.$$

Where $h(T-t)$ is defined in (8).

Remark: The optimal strategies for the diffusion model depends only on the safety loading θ, time $T-t$, the expected instantaneous rate of return of the risky asset r_1, the volatility of the risky asset price σ_1 and interest rate r. That is, the claim size distribution as well as the counting process have no effect on the optimal strategies.

References

[1] Hipp C, Taksar M. Stochastic control for optimal new business[J]. Insurance: Mathematics and Economics, 2000, 26(2): 185-192.

[2] Hipp C, Plum M. Optimal investment for insurers[J]. Insurance: Mathematics and Economics, 2000, 27(2): 215-228.

[3] Browne S. Optimal investment policies for a firm with a random risk process:exponential utility and minimizing and probability of ruin [J]. Mathematics of Operations Research, 1995, 20(4): 937-957.

[4] Yang H L, Zhang L H. Optimal investment for insurer with jump diffusion risk process[J]. Insurance: Mathematics and Economics, 2005, 37(3): 615-634.

[5] Liang Z, Bayraktar E. Optimal reinsurance and investment with unobservable claim size and intensity. Insurance: Mathematics and Economics, 2014, 55: 156-166.

[6] Huang Y, Yang X, Zhou J. Optimal investment and proportional reinsurance for a jump diffusion risk model with constrained control variable. J.Comput.Appl.Math. 2016, 296: 443-461.

[7] Bai Li hua, Cai Jun, Zhou Ming. Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting[J]. Insurance: Mathematics and Economics, 2013, 53(3): 664-670.

[8] Liang Zhi bing, Yuen K C. Optimal dynamic reinsurance with dependent risks: variance premium principle[J]. Scandinavian Actuarial Journal, 2016. 2016(1): 18-36.

[9] Yuen K C, Liang Zhi bing, Zhou ming. Optimal proportional reinsurance with common shock dependence[J]. Insurance: Mathematics and Economics, 2015, 64(1): 1-13.

[10] Liang Z B. Optimal investment and optimal reinsurance for jump diffusion surplus process[J]. Journal of Mathematics, 2008, 51(6): 1195-1204.

[11] Lin X, Li Y F. Optimal investment and reinsurance strategies for jump diffusion risk models[J]. Journal of Applied Mathematics, 2013, 36(05): 791-802.

[12] Wang Ym, Wang D, Long M. Optimal investment and reinsurance strategies under jump diffusion risk model[J]. Journal of Zhengzhou University (Science Edition), 2015, 47(01): 50-54.

[13] Xiao H M, Li H. Ruin probability of delayed claims risk model with constant interest force under heavy tailed claims[J]. Journal of Northwest Normal University(Natural Science Edition), 2011, 47(06): 17-19+29

[14] Xiao H M, Liu A L, He Y. The limit property of the delay risk model with dependent compensation and investment[J]. Journal of Lanzhou University (Natural Science Edition), 2017, 53(05): 696-700+705.

[15] Goovareta M J, Kass R, Van Heerwarrden A N. Bauwelincks T. Effective Actuarial Methods[J]. North Holland: Amsterdam, 1990.

[16] Fleming W H, Soner H M. Controlled Markov processes and viscosity solutions[M]. Springer Science and Business Media, 2006.

[17] Schmidli H, Stochastic Control in Insurance[M]. London: Springer-Verlag, 2008.

[18] Yang Hai liang, Zhang Li hong. Optimal investment for insurer with jump diffusion risk process[J]. Insurance: Mathematics and Economics, 2005, 37(3): 615-634