Existence and multiplicity of positive solutions of second-order three-point boundary value problems

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***Abstract*—In this paper,we study the existence and multiplicity of positive solutions of second-order three-point boundary value problems**

$$\begin{cases} u''(t)+a(t)f(u(t))=0, & t\in(0,1), \\ u(0)=0, u(1)=\alpha(u(\eta))u(\eta), \end{cases}$$

where f**:**$[0,\infty)\to[0,\infty)$ **is continuous,** $0<\eta<1$**,** $\alpha_1\le\alpha\le\alpha_2$**,** $0<\eta\alpha(s)<1$**,** $s\in R^+$ **,** α_1**,** α_2 **is a constant.** $a:[0,1]\to[0,\infty)$ **and** $\exists\, x_0\in[\eta,1]$ **such that** $a(x_0)>0$**. The proof of the main results is based on the fixed point theorem in cones.**

***Index Terms*—Three-point boundary value problem; Positive solutions; Fixed point theorem in cones; Existence**
***MSC(2010):*—39A10, 39A12**

I. INTRODUCTION

The study of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev[7-8]. Then Gupta [5] studied three-point boundary value problems for nonlinear differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by several authors by several authors by using the Leray-Schauder Continuation Theorem,Nonlinear Alternatives of Leray-Schauder, and coincidence degree theory .We refer the reader to [1-3,6,10-12]for some recent results of nonlinear multi-point boundary value problems.

In this paper, we consider the existence of positive solutions to the equation

$$\begin{cases} u''(t)+a(t)f(u(t))=0, & t\in(0,1), \\ u(0)=0, u(1)=\alpha(u(\eta))u(\eta), \end{cases} \quad (1.1)$$

where $0<\eta<1$, Our purpose here is to give some existence results for positive solutions to (1.1) ,assuming that $\alpha\eta<1$ and f is either superlinear or sublinear. Our proof is based upon the fixed point theorem in a cone.

From now on, we make the following assumptions:

$(H1)$ f:$[0,\infty)\to[0,\infty)$ is continuous;

$(H2)$ $a:[0,1]\to[0,\infty)$ and $\exists\, x_0\in[\eta,1]$ such that $a(x_0)>0$.

Set

$$f_0=\lim_{u\to0^+}\frac{f(u)}{u},\ f_\infty=\lim_{u\to\infty}\frac{f(u)}{u},$$

then $f_0=0$ and $f_\infty=\infty$ correspond to the superlinear case, and $f_\infty=0$correspond to the sublinear case. By the positive solution of (1.1) we understand a function $u(t)$ which is positive on $0<t<1$ and satisfies the differential equation (1.1) .

The main results of the present paper are as follows:

Theorem 1. Let $(H1)$ - $(H2)$ hold. Then the problem (1.1) has at least one positive solution in the case

(i) $f_0=0$ and $f_\infty=\infty$ (superlinear)or

(ii) $f_0=\infty$ and $f_\infty=0$ (sublinear).

The proof of above theorem is based upon an application of the following well-known Guo's fixed point theorem[4]

Theorem 2. Let E be a Banach space, and let $K\subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let

$$A:\ K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$$

be a completely continuous operator such that

(i) $\|Au\|\le\|u\|$, $u\in K\cap\partial\Omega_1$, and $\|Au\|\ge\|u\|$, $u\in K\cap\partial\Omega_2$; or

(ii) $\|Au\|\ge\|u\|$, $u\in K\cap\partial\Omega_1$, and $\|Au\|\le\|u\|$, $u\in K\cap\partial\Omega_2$.

Then A has a fixed point in $K\cap(\overline{\Omega}_2\cap\Omega_1)$.

II. PRELIMINARIES

$C[0,1]$ is a Banach space. The norm in $C[0,1]$ is defined as follows

$$|u|_0=\max_{t\in[0,1]}|u(t)|.$$

Lemma1. Let $\alpha(u(\eta))\eta\ne1$ then for $y\in C[0,1]$, the problem

$$\begin{cases} u''(t)+y(t)=0, & t\in(0,1), \\ u(0)=0, u(1)=\alpha(u(\eta))u(\eta), \end{cases} \quad (2.1)$$

has a unique solution

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$$u(t)=\int_0^1 G(t,s)f(y(s))ds$$
$$+\frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds.$$
$$:=Au(t),t\in(0,1).$$

Where

$$H(t,s)=G(t,s)+\frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}G(\eta,s).\ (2.2)$$

And

$$G(t,s)=\begin{cases}t(1-s),0\le t\le s\le 1,\\ s(1-t),0\le s\le t\le 1,\end{cases}$$
$$G(\eta,s)=\begin{cases}\eta(1-s),\eta\le s\le 1,\\ s(1-\eta),0\le s\le \eta.\end{cases}$$

Lemma 2. Let $0<\alpha(\mathrm{u}(\eta))<\frac{1}{\eta}$. If $y\in C[0,1]$ and $y\ge 0$,then the unique solution u of the problem (1.1) satisfies

$$u\ge 0,t\in[0,1].$$

Proof From the fact that $u''(x)=-y(x)\le 0$,we know that the graph of $u(t)$ is concave down on $(0,1)$. So if $u(1)\ge 0$ then the concavity of u and the boundary condition $u(0)=0$, imply that $u\ge 0$ for $t\in[0,1]$.

If $u(1)<0$,then we have that

$$u(\eta)<0,\ (2.3)$$

and

$$u(1)=\alpha(u(\eta))u(\eta)>\frac{1}{\eta}u(\eta)\ (2.4)$$

This contradicts the concavity of u .

Lemma 3. Let $\alpha(u(\eta))\eta>1$. If $y\in C[0,1]$ and for $y\ge 0$, then the problem (1.1) has no positive solution.

Proof Assume that has a positive solution u

If $u(1)>0$,then $u(\eta)>0$, and

$$\frac{u(1)}{1}=\frac{\alpha(u(\eta))u(\eta)}{1}>\frac{u(\eta)}{\eta},\ (2.5)$$

this contradicts the concavity of u .

If $u(1)=0$ and for some $\tau\in(0,1)$, $u(\tau)>0$ then

$$u(\eta)=u(1)=0,\ \tau\ne\eta\ (2.6)$$

If $\tau\in(0,\eta)$, then $u(\tau)>u(\eta)=u(1)$, which contradicts the concavity of u . If $\tau\in(\eta,1)$, then $u(0)=u(\eta)<u(\tau)$, which contradicts the concavity of u again.

In the rest of the paper, we assume that $\alpha(u(\eta))\eta<1$.

Lemma 4. Let $0<\alpha(u(\eta))<\frac{1}{\eta}$ .If $y\in C[0,1]$ and $y\ge 0$, then the unique solution of the problem (1.1) satisfies

$$\min_{t\in[\eta,1]}u(t)\ge\gamma\|u\|$$

Where $\gamma=\min\{\alpha_1\eta,\frac{\alpha_1(1-\eta)}{1-\alpha_1\eta},\eta\}$.

Proof We divide the proof into two steps.

Step1. We deal with the case $0<\alpha(u(\eta))<1$.

In this case, by Lemma 2 ,we know that

$$u(\eta)\ge u(1).\ (2.7)$$

Set

$$u(\bar t)=\|u\|.\ (2.8)$$

If $\bar t\le\eta<1$, then

$$\min_{t\in[\eta,1]}u(t)=u(1),\ (2.9)$$

and

$$u(\bar t)\le u(1)+\frac{u(1)-u(\eta)}{1-\eta}(0-1)$$
$$=u(1)[1-\frac{1-\frac{1}{\alpha}}{1-\eta}]$$
$$=u(1)\frac{1-\alpha\eta}{\alpha\ (1-\eta)}$$
$$\le u(1)\frac{1-\alpha_1\eta}{\alpha_1(1-\eta)}$$

This together with (2.9) implies that

$$\min_{t\in[\eta,1]}u(t)\ge\frac{\alpha_1(1-\eta)}{1-\alpha_1\eta}\|u\|\ (2.10).$$

If $\eta<\bar t<1$, then

$$\min_{t\in[\eta,1]}u(t)=u(1),(2.11)$$

From the concavity of u , we know that

$$\frac{u(\eta)}{\eta}\ge\frac{u(\bar t)}{\bar t}.(2.12)$$

Combining (2.12) and boundary condition $\alpha(u(\eta))u(\eta)=u(1)$, we conclude that

$$\frac{u(1)}{\alpha(u(\eta))\eta}\ge\frac{u(\bar t)}{\bar t}\ge u(\bar t)=\|u\|,$$

This is

$$\min_{t\in[\eta,1]}u(t)\ge\alpha(u(\eta))\eta\|u\|\ge\alpha_1(u(\eta))\eta\|u\|.\ (2.13)$$

Step 2. We deal with the case $1\le\alpha(u(\eta))<\frac{1}{\eta}$. In this case , we have

$$u(\eta)\le u(1).(2.14)$$

Set

$$u(\bar t)=\|u\|,(2.15)$$

then we can choose $\bar t$ such that

$$\eta\le\bar t\le 1.(2.16)$$

(we note that if $\bar t\in[0,1]\backslash[\eta,1]$, then the point $(\eta,u(\eta))$

is below the straight line determined by $(1,u(1))$ and $(\bar{t},u(\bar{t}))$. This contradicts the concavity of u). From (1.16) and the concavity of u, we know that

$$\min_{t\in[\eta,1]} u(t)=u(\eta).(2.17)$$

Using the concavity of u and Lemma 2, we have that

$$\frac{u(\eta)}{\eta}\le\frac{u(\bar{t})}{\bar{t}} \quad (2.18)$$

This implies

$$\min_{t\in[\eta,1]} u(t)\ge\eta\|u\|.(2.19)$$

This completes the proof.

III. PROOF OF THE MAIN RESULT

Proof of Theorem 1 Superlinear case. Suppose then that $f_0=0$ and $f_\infty=\infty$. We wish to show the existence of a positive solution of (1.1) .Now (1.1) has a solution $y=y(t)$ if and only if y solves the operator equation

$$y(t)=\int_0^1 G(t,s)f(y(s))ds+\frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$:=Ay(t) \quad (3.1)$$

Denote

$$K=\{y\mid y\in C[0,1],\ y\le 0,\min_{\eta\le t\le 1} y(t)\ge\gamma\|y\|\}\ (3.2)$$

It is obvious that K is a cone in $C[0,1]$ .Moreover, by Lemma 4, It is also easy to check that $A:K\to K$ is completely continuous.

Now since $f_0=0$, we may choose $H_1>0$ so that $f(y)\le\varepsilon y$, for $0<y\le H_1$ where $\varepsilon>0$ satisfies

$$\varepsilon[1+\frac{\alpha_2(u(\eta))}{1\text{-}\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)\mathrm{ds}\le 1. \qquad (3.3)$$

Thus, if $y\in K$ and $\|y\|=H_1$,then from (3.1) and (3.3) ,we get

$$Ay(t)\le\int_0^1 G(s,s)f(y(s))ds+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)f(y(s))ds$$
$$\le\int_0^1 G(s,s)\varepsilon y(s)ds+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)\varepsilon y(s)ds$$
$$\le\varepsilon\int_0^1 G(s,s)\|y\|ds+\frac{\varepsilon\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)\|y\|ds$$
$$\le\varepsilon\int_0^1 G(s,s)dsH_1+\frac{\varepsilon\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)dsH_1$$
$$\le\varepsilon[1+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)dsH_1 \qquad (3.4)$$

Now if we let

$$\Omega_1=\{y\in C[0,1]\mid \|y\|<H_1\},\ (3.5)$$

then (3.4) show that $\|Ay\|\le\|y\|$, for $y\in K\cap\partial\Omega_1$.

Further, since $f_\infty=\infty$, there exists $\hat{H}_2>0$ such that $f(u)\ge\rho u$, for $u\ge\hat{H}_2$,where $\rho>0$ is chosen so that

$$\frac{\rho\gamma\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)ds\|y\|\ge 1.(3.6)$$

Let $H_2=\max\{2H_1,\frac{\hat{H}_2}{\gamma}\}$ and $\Omega_2=\{y\in C[0,1]\mid \|y\|<H_2\}$, then $y\in K$ and $\|y\|=H_2$ implies

$$\min_{\eta\le t\le 1} y(t)\ge\gamma\|y\|\ge\hat{H}_2,$$

and so

$$Ay(\eta)=\int_0^\eta G(\eta,s)f(y(s))dt+\frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$\ge-\frac{\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$\ge\frac{\rho\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$\ge\frac{\rho\gamma\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)ds\|y\|\ \ (by\,\eta<1)\quad(3.7)$$

Hence, for $y\in K\cap\partial\Omega_2$

$$\|Ay\|\ge\frac{\rho\gamma\alpha_1(u(\eta))}{1-\alpha\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)a(s)ds\|y\|$$
$$\ge\|y\|.$$

Therefore, by the first part of the Fixed Point Theorem, it follows that A has a fixed point in $K\cap(\overline{\Omega}_2\setminus\Omega_1)$, such that $H_1\le\|u\|\le H_2$. This completes the superlinear part of the theorem.

Sublinear case. Suppose next that $f_0=\infty$ and $f_\infty=0$. We first choose $H_3>0$ such that $f(y)\ge My$ for $0<y<H_3$, where

$$\frac{M\gamma\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)ds\ge 1\,(3.8)$$

By using the method to get (3.7) ,we can get that

$$Ay(\eta)=\int_0^1 G(\eta,s)f(y(s))ds+\frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$\ge\frac{\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$
$$\ge\frac{\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)M(y(s))ds$$
$$\ge\frac{\alpha_1(u(\eta))}{1-\alpha_1(u(\eta))\eta}\int_0^1 G(\eta,s)ds\|y\|$$
$$\ge H_3.\quad(3.9)$$

Thus we may let $\Omega_3=\{y\in C[0,1]\mid \|y\|<H_3\}$ so that

$$\|Ay\|\ge\|y\|,\ y\in K\cap\partial\Omega_3.$$

Now, since $f_\infty=0$, there exists $\hat{H}_4>0$ so that $f(y)\le\lambda y$ for $y\ge\hat{H}_4$, where $\lambda>0$ satisfies

$$\lambda[1+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)ds \le 1. \quad (3.10)$$

We consider two cases:

Case(i). Suppose f is bounded, say $f(y) \le N$ for all $y \in [0,\infty)$. In this case choose

$$H_4 = \max\{2H_3, N[1+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)ds\},$$

so that for $y \in K$ with $\|y\| = H_4$ we have

$$Ay(t) = \int_0^1 G(t,s)f(y(s))ds + \frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$

$$\le \int_0^1 G(s,s)Nds + \frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)Nds$$

$$\le N[1+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)ds$$

$$\le H_4$$

and therefore $\|Ay\| \le \|y\|$.

Case(ii). If f is unbounded, then we know from $(A1)$ that there is $H_4 : H_4 > \max\{2H_3, \frac{1}{\lambda}\hat{H}_4\}$ such that

$$f(y) \le f(H_4) \text{ for } 0 < y \le H_4.$$

(We are able to do this since f is unbounded). Then for $y \in K$ and $\|y\| = H_4$ we have

$$Ay = \int_0^1 G(t,s)f(y(s))ds + \frac{\alpha(u(\eta))}{1-\alpha(u(\eta))\eta}\int_0^1 G(\eta,s)f(y(s))ds$$

$$\le \int_0^1 G(s,s)f(y(s))ds + \frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}\int_0^1 G(s,s)f(H_4)ds$$

$$\le \lambda H_4[1+\frac{\alpha_2(u(\eta))}{1-\alpha_2(u(\eta))\eta}]\int_0^1 G(s,s)ds$$

$$\le H_4,$$

Therefore, in either case we may put

$$\Omega_4 = \{y \in C[0,1] \mid \|y\| < H_4\},$$

and for $y \in K \cap \partial\Omega_4$ we may have $\|Ay\| \le \|y\|$. By the second part of the Fixed Point Theorem, it follows that (1.1) has a positive solution. Therefore, we have completed the proof of Theorem 1.

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