STRONG CONVERGENCE OF A UNIFORM KERNEL ESTIMATOR FOR INTENSITY OF A PERIODIC POISSON PROCESS WITH UNKNOWN PERIOD

I WAYAN MANGKU

Abstract. Strong convergence of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. The result presented here is a special case of the one in [3]. The aim of this paper is to present an alternative and a relatively simpler proof of strong convergence compared to the one in [3]. This is a joint work with R. Helmers and R. Zitikis.

1991 Mathematics Subject Classification: 60G55, 62G05, 62G20.

Keywords and Phrases: periodic Poisson process, intensity function, uniform kernel estimator, strong convergence.

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

1. Introduction and main result

In this paper, strong convergence of a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and proved. For more general results which using general kernel function can be found in [3] and chapter 3 of [4].

Let X be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with unknown period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal is: (a) To present a uniform kernel estimator for λ at a given point $s \in [0, n]$ using only a single realization $X(\omega)$ of the Poisson process X observed in interval $[0, n]$. (The requirement $s \in [0, n]$ can be dropped if we know the period τ.) (b) To determine an alternative set of conditions for having strong convergence of this estimator compared to the one in [3]. (c) To present an alternative and a relatively simpler proof of strong convergence of the estimator compared to the one in [3].

Note that, since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, n]$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

We will assume throughout that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)| dx = 0$$

(e.g. [7], p.107-108). This assumption is a mild one since the set of all Lebesgue points of λ is dense in $\mathbf{R}$, whenever λ is assumed to be locally integrable.

Let $\hat{\tau}_n$ be any consistent estimator of the period τ, that is,

$$\hat{\tau}_n \xrightarrow{p} \tau,$$

as $n \to \infty$. For example, one may use the estimators constructed in [2] or perhaps the estimator investigated by [6] or [1]. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{1.2}$$

as $n \to \infty$. With these notations, we may define an estimator of $\lambda(s)$ as

$$\hat{\lambda}_n(s) := \frac{\hat{\tau}_n}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n - h_n] \cap [0, n]\right). \tag{1.3}$$

The idea behind the construction of the estimator $\hat{\lambda}_n(s)$ given in (1.3) can be found e.g. in [5].

The main result of this paper is the following theorem.

Theorem 1.1. *Let the intensity function λ be periodic and locally integrable. Furthermore, let the bandwidth h_n be such that (1.2) holds true, and*

$$\frac{1}{nh_n} = \mathcal{O}(n^{-\alpha}) \tag{1.4}$$

and

$$n|\hat{\tau}_n - \tau|/h_n = \mathcal{O}(n^{-\beta}) \tag{1.5}$$

with probability 1, as $n \to \infty$, for an arbitrarily small $\alpha > 0$ and $\beta > 0$, then

$$\hat{\lambda}_n(s) \stackrel{a.s.}{\to} \lambda(s) \tag{1.6}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_n(s)$ converges strongly to $\lambda(s)$ as $n \to \infty$.

2. Proofs of Theorem 1.1

Throughout this paper, for any random variables Y_n and Y on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, we write $Y_n \stackrel{c}{\to} Y$ to denote that Y_n *converges completely* to Y, as $n \to \infty$. We say that Y_n converges completely to Y if

$$\sum_{n=1}^{\infty} \mathbf{P}(|Y_n - Y| > \epsilon) < \infty,$$

for every $\epsilon > 0$.

Let $B_h(x)$ denotes the interval $[x-h, x+h]$. To establish Theorem 1.1, first we prove

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) \stackrel{a.s.}{\to} \lambda(s), \tag{2.1}$$

as $n \to \infty$, where $N_n = \#\{k : \ s + k\tau \in [0,n]\}$. To prove (2.1), by Borel-Cantelli, it suffices to check, for each $\epsilon > 0$, that

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\left|\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - \lambda(s)\right| > \epsilon\right) < \infty, \tag{2.2}$$

i.e. the difference between the quantity on the l.h.s. of (2.1) and $\lambda(s)$ converges completely to zero, as $n \to \infty$. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we obtain (2.2), which implies (2.1).

Then, to prove (1.6), it remains to check that $\hat{\lambda}_n(s)$ can be replaced by the quantity on the l.h.s. of (2.1), i.e. we must show that the difference between $\hat{\lambda}_n(s)$ and the quantity on the l.h.s. of (2.1) converges almost surely to zero, as $n \to \infty$. To show this, first we write this difference as

$$\left(\frac{\hat{\tau}_n N_n}{n} - 1\right) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right), \tag{2.3}$$

that is, the quantity on the l.h.s. of (2.1) multiplied by $(\hat{\tau}_n N_n n^{-1} - 1)$. Since $\lambda(s)$ is finite, by (2.1), we have that the quantity on the l.h.s.

of (2.1) is $\mathcal{O}(1)$, with probability 1, as $n \to \infty$. Hence, it remains to check that

$$\left|\frac{\hat{\tau}_n N_n}{n} - 1\right| = o(1), \tag{2.4}$$

with probability 1, as $n \to \infty$. By the triangle inequality, the quantity on the l.h.s. of (2.4) does not exceed

$$\left|\frac{\hat{\tau}_n N_n}{n} - \frac{\hat{\tau}_n}{\tau}\right| + \left|\frac{\hat{\tau}_n}{\tau} - 1\right| \leq \frac{\hat{\tau}_n}{n}\left|N_n - \frac{n}{\tau}\right| + \frac{1}{\tau}\left|\hat{\tau}_n - \tau\right|. \tag{2.5}$$

Note that $|n/\tau - N_n| \leq 1$, and $\hat{\tau}_n = \mathcal{O}(1)$, with probability 1, as $n \to \infty$ (by (1.5)). Hence, the first term on the r.h.s. of (2.5) is $\mathcal{O}(n^{-1})$, with probability 1, as $n \to \infty$. By (1.5), we have that its second term is $o(1)$, with probability 1, as $n \to \infty$. Therefore we have (2.4). This completes the proof of Theorem 1.1.

In the following lemma we shall show that we may replace the random centre $s + k\hat{\tau}_n$ of the interval $B_{h_n}(s + k\hat{\tau}_n)$ in (2.1) by its deterministic limit $s + k\tau$.

Lemma 2.1. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.5) are satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0,n]\right)\right\}\right| \xrightarrow{c} 0, \tag{2.6}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: First note that the difference within curly brackets on the l.h.s. of (2.6) does not exceed

$$X\left(B_{h_n}(s + k\hat{\tau}_n) \Delta B_{h_n}(s + k\tau) \cap [0,n]\right). \tag{2.7}$$

Now we notice that

$$B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \subseteq B_{h_n}(s + k\hat{\tau}_n) \subseteq B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau). \tag{2.8}$$

By (2.7) and (2.8) we have

$$\begin{aligned} &\left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0,n]\right)\right\}\right| \\ &\leq 2X\left(B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau) \setminus B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \cap [0,n]\right). \end{aligned} \tag{2.9}$$

Hence, to prove (2.6), it suffices to show that

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\cap[0,n]\right) \stackrel{c}{\to} 0, \tag{2.10}$$

as $n\to\infty$. To prove (2.10) we argue as follows. Let Λ_n denotes the l.h.s. of (2.10), and let also $\epsilon>0$ be any fixed real number. Then to verify (2.10) it suffices to check, for each $\epsilon>0$,

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|\Lambda_n|>\epsilon\right)<\infty. \tag{2.11}$$

By the assumption (1.5), there exists large fixed positive integer n_0 and posistive constant C such that $n|\hat{\tau}_n-\tau|\leq Cn^{-\beta}h_n$ with probability 1, for all $n\geq n_0$. Then, for all $n\geq n_0$, we have with probability 1 that $\mathbf{P}(|\Lambda_n|>\epsilon)\leq\mathbf{P}(|\bar{\bar{\Lambda}}_n|>\epsilon)$, where $\bar{\bar{\Lambda}}_n$ is given by

$$\bar{\bar{\Lambda}}_n = \frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n} X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap[0,n]\right). \tag{2.12}$$

(Note that $\bar{\bar{\Lambda}}_n$ is precisely equal to $\bar{\Lambda}_n$ in (2.10), provided we replace, for our present purposes, δ by $Cn^{-\beta}$). Since to show convergency of an infinite series it suffices to check convergency of its tail, to prove (2.11), it suffices to check, for each $\epsilon>0$, that

$$\sum_{n=n_0}^{\infty}\mathbf{P}\left(|\bar{\bar{\Lambda}}_n|>\epsilon\right)<\infty. \tag{2.13}$$

By Markov inequality for the M-th moment, we then obtain

$$\mathbf{P}\left(|\bar{\bar{\Lambda}}_n|>\epsilon\right)\leq\frac{E(\bar{\bar{\Lambda}}_n)^M}{\epsilon^M}=\left(\frac{1}{2\epsilon N_n h_n}\right)^M \mathbf{E}\left(\sum_{k=-\infty}^{\infty}X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap W_n\right)\right)^M. \tag{2.14}$$

Now consider the expectation on the r.h.s. of (2.14). By writing the M-th power of a sum as a M-multiple sum, we can interchange summations and expectation. Note that for large n, by (1.2), the random variables

$X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\right)$ and

$$X\left(B_{h_n(1+Cn^{-\beta})}(s+j\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+j\tau)\right)$$

for $k\neq j$, are independent. Now, we distinguish M different cases in the M-multiple sum, namely, case (1) if all indexes are the same, up to case (M) if all indexes are different. Then we split up the M-multiple sum into M different terms, where each term corresponds to each of the M cases. Because for each $k\in\mathbb{Z}$ and for any fixed M, by (1.2), it is easy to check that

$$\mathbf{E}\left(X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\right)\right)^M=O(1), \quad (2.15)$$

as $n\to\infty$, uniformly in k, we find that for large n, the biggest term among those M terms, is the term corresponds to the case where all indexes are different. Hence we conclude that the expectation on the r.h.s. of (2.14) does not exceed

$$\begin{aligned}
&M\left(\sum_{k=-\infty}^{\infty}\mathbf{E}X\left(B_{h_n(1+Cn^{-\beta})}(s+k\tau)\setminus B_{h_n(1-Cn^{-\beta})}(s+k\tau)\cap W_n\right)\right)^M\\
&=M\left(\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}\lambda(s+x)\right.\\
&\left.\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in W_n)dx\right)^M\\
&\leq M\left(N_n+1\right)^M\left(\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}\lambda(s+x)dx\right)^M.
\end{aligned} \quad (2.16)$$

The integral on the r.h.s. of (2.16) does not exceed

$$\begin{aligned}
&\int_{B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)}|\lambda(s+x)-\lambda(s)|dx\\
&+\left|B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)\right|\lambda(s).
\end{aligned} \quad (2.17)$$

Since s is a Lebesgue point of λ, we have that the quantity in the first term of (2.17) is of order $o(n^{-\beta}h_n)$, as $n\to\infty$. Since $\lambda(s)$ is finite and $\left|B_{(1+Cn^{-\beta})h_n}(0)\setminus B_{(1-Cn^{-\beta})h_n}(0)\right|=4Cn^{-\beta}h_n$, we have that the quantity in the second term of (2.17) is of order $O(n^{-\beta}h_n)$, as $n\to\infty$. Hence, the r.h.s. of (2.16) is of order $O\left(n^{M(1-\beta)}h_n^M\right)$, which implies that the r.h.s. of (2.14) is of order $O\left(n^{-M\beta}\right)$, as $n\to\infty$. By choosing $M>\frac{1}{\beta}$, we see that (2.13) is proved. This completes the proof of Lemma 2.1.

To complete our proof of Theorem 1.1 we also need the following lemma.

Lemma 2.2. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.4) are satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\left|X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)-\mathbf{E}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)\right| \stackrel{c}{\to} 0, \tag{2.18}$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: First we write the l.h.s. of (2.18) as

$$\frac{1}{2N_nh_n}\left|\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right|, \tag{2.19}$$

where we write $\tilde{X}$ to denote $X-\mathbf{E}X$. By Markov inequality for the $2M$-th moment, for each $\epsilon>0$, we then obtain

$$\mathbf{P}\left(\frac{1}{2N_nh_n}\left|\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right|>\epsilon\right) \leq \left(\frac{1}{2\epsilon N_nh_n}\right)^{2M}\mathbf{E}\left(\sum_{k=-\infty}^{\infty}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^{2M}. \tag{2.20}$$

Now consider the expectation on the r.h.s. of (2.20). By writing the $2M$-th power of a sum as a $2M$-multiple sum, we can interchange summations and expectation. For large n, the r.v. $X\left(B_{h_n}(s+k\tau)\cap W_n\right)$ and $X\left(B_{h_n}(s+j\tau)\cap W_n\right)$, for $k\neq j$, are independent. Here we also distinguish $2M$ different cases in the $2M$-multiple sum, namely, case (1) if all indexes are the same, up to case $(2M)$ if all indexes are different. Then we also split up the $2M$-multiple sum into $2M$ different terms, where each term corresponds to each of the $2M$ cases. Because for any fixed M, it is easy to check that $\mathbf{E}\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)=0$ and $\mathbf{E}\left(\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^{2M}=O(1)$ as $n\to\infty$, uniformly in k, we find for large n, the biggest term among those $2M$ terms, is the one corresponds to the case where there are M pairs of the same indexes. Hence we conclude that the expectation on the r.h.s. of (2.20) does not exceed

$$
\begin{aligned}
&2M\left(\sum_{k=-\infty}^{\infty}\mathbf{E}\left(\tilde{X}\left(B_{h_n}(s+k\tau)\cap W_n\right)\right)^2\right)^M\\
&= M2^{M+1}h_n^M\left(\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{B_{h_n}(0)}\lambda(s+x)\mathbf{I}\left(s+k\tau+x\in W_n\right)dx\right)^M\\
&\le M2^{M+1}h_n^M(N_n+1)^M\left(\frac{1}{2h_n}\int_{B_{h_n}(0)}\lambda(s+x)dx\right)^M\\
&= O(n^M h_n^M), \qquad (2.21)
\end{aligned}
$$

as $n\to\infty$. Combining this result with the assumption (1.4), we then obtain that the r.h.s. of (2.20) is of order $O\left(n^{-M}h_n^{-M}\right) = O\left(n^{-M\alpha}\right)$, as $n\to\infty$. By choosing $M > \frac{1}{\alpha}$, we have that the probabilities on the l.h.s. of (2.20) are summable, which implies this lemma. This completes the proof of Lemma 2.2.

It remains to evaluate a non-random sum.

Lemma 2.3. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) is satisfied, then*

$$
\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\mathbf{E}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right) = \lambda(s)+o(1), \qquad (2.22)
$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: Using the fact that X is Poisson, the l.h.s. of (2.22) can be written as

$$
\begin{aligned}
&\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx\\
&= \frac{1}{2N_nh_n}\int_{-h_n}^{h_n}\lambda(s+x)\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])dx. \qquad (2.23)
\end{aligned}
$$

Now note that

$$
(N_n-1)\le\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\le(N_n+1),
$$

which implies $N_n^{-1}\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])$ can be written as $(1+\mathcal{O}(n^{-1}))$, as $n\to\infty$, uniformly in x. Then, the quantity on the r.h.s. of (2.23) can be written as

$$
\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx. \qquad (2.24)
$$

By (1.2) together with the assumption that s is a Lebesgue point of λ, we have that

$$(2h_n)^{-1}\int_{-h_n}^{h_n}\lambda(s+x)dx=\lambda(s)+o(1),$$

as $n\to\infty$. Then we obtain this lemma. This completes the proof of Lemma 2.3.

Lemma 2.4. *Suppose that the assumption (1.5) is satisfied. Then, for each positive integer M, we have that*

$$\mathbf{E}\left(\hat{\tau}_n-\tau\right)^{2M}=O\left(n^{-2M(1+\beta)}h_n^{2M}\right), \qquad (2.25)$$

as $n\to\infty$.

Proof: By the assumption (1.5), there exists large positive constant C and positive integer n_0 such that

$$|\hat{\tau}_n-\tau|\le Cn^{-(1+\beta)}h_n, \qquad (2.26)$$

with probability 1, for all $n\ge n_0$. Then, the l.h.s. of (2.25) can be written as

$$\begin{aligned}&\int_0^\infty x^{2M}d\mathbf{P}\left(|\hat{\tau}_n-\tau|\le x\right)\\ &=-\int_0^{Cn^{-(1+\beta)}h_n}x^{2M}d\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right). \qquad (2.27)\end{aligned}$$

By partial integration, the r.h.s. of (2.27) is equal to

$$\begin{aligned}&-x^{2M}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)\Big|_0^{Cn^{-(1+\beta)}h_n}\\ &+2M\int_0^{Cn^{-(1+\beta)}h_n}\mathbf{P}\left(|\hat{\tau}_n-\tau|>x\right)x^{2M-1}dx. \qquad (2.28)\end{aligned}$$

The first term of (2.28) is equals to zero, while its second term is at most equal to

$$\begin{aligned}2M\int_0^{Cn^{-(1+\beta)}h_n}x^{2M-1}dx &= C^{2M}n^{-2M(1+\beta)}h_n^{2M}\\ &= O\left(n^{-2M(1+\beta)}h_n^{2M}\right), \qquad (2.29)\end{aligned}$$

as $n\to\infty$. This completes the proof of Lemma 2.4.

References

[1] M. Bebbington and R. Zitikis (2004). A robust heuristic estimator for the period of a Poisson intensity function. *Methodology and Computing in Applied Probability*, **6**, 441-462.

[2] R. Helmers and I W. Mangku (2003), On estimating the period of a cyclic Poisson process. *Mathematical Statistics and Applications: Festschrift in honor of Constance van Eeden.* (Editors: Marc Moore, Sorana Froda and Christian Leger), IMS Lecture Notes Series - Monograph Series, Volume **42**, 345-356.

[3] R. Helmers, I W. Mangku, and R. Zitikis (2003), Consistent estimation of the intensity function of a cyclic Poisson process. *J. Multivariate Anal.* **84**, 19-39.

[4] Mangku, I W. (2001). *Estimating the Intensity of a Cyclic Poisson Process*, Ph.D Thesis, University of Amsterdam, The Netherlands.

[5] Mangku, I W. (2008). Consistency of a uniform kernel estimator for intensity of a periodic Poisson process with unknown period. *Journal of Mathematics and Its Applications,* **7**, 2, 31-37.

[6] D. Vere-Jones (1982). On the estimation of frequency in point-process data. *J. Appl. Probab.* **19A**, 383-394.

[7] R. L. Wheeden and A. Zygmund (1977), *Measure and Integral: An Introduction to Real Analysis.* Marcel Dekker, Inc., New York.