CONSISTENCY OF A UNIFORM KERNEL ESTIMATOR FOR INTENSITY OF A PERIODIC POISSON PROCESS WITH UNKNOWN PERIOD

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Abstract. A uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and a proof of its consistency is discussed. The result presented in this paper is a special case of that in [3]. The aim of discussing a uniform kernel estimator is in order to be able to present a relatively simpler proof of consistency compared to that in [3]. This is a joint work with R. Helmers and R. Zitikis.

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1. Introduction and main result

In this paper, a uniform kernel estimator for intensity of a periodic Poisson process with unknowm period is presented and a proof of its consistency is discussed. The result presented here is a special case of that in [3] and chapter 3 of [5].

Let X be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with unknown period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all $k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson process X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal in this paper is: (a) To study construction of a uniform kernel estimator for λ at a given point $s \in [0, n]$ using only a single realization $X(\omega)$ of the Poisson process X observed in interval $[0, n]$. (The requirement $s \in [0, n]$ can be dropped if we know the period τ.) (b) To determine the minimal conditions for having weak convergence of this estimator.

Note that, since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, n]$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we assume that $s \in [0, \tau)$.

Note also that, the meaning of the asymptotic $n \to \infty$ in this paper is somewhat different from the classical one. Here n does not denote our sample size, but it denotes the length of the interval of observations. The size of our samples is a random variable denoted by $X([0, n])$.

Let $\hat{\tau}_n$ be any consistent estimator of the period τ, that is, $\hat{\tau}_n \xrightarrow{p} \tau$, as $n \to \infty$. For example, one may use the estimators constructed in [2] or perhaps the estimator investigated by [6] and [1]. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0 \tag{1.2}$$

as $n \to \infty$. With these notations, we now define an estimator of $\lambda(s)$ as

$$\hat{\lambda}_n(s) := \frac{\hat{\tau}_n}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left([s + k\hat{\tau}_n - h_n, s + k\hat{\tau}_n - h_n] \cap [0, n]\right). \tag{1.3}$$

Let us now describe the idea behind the construction of the estimator $\hat{\lambda}_n(s)$. Note that, since there is only one realization of the Poisson process X available, we have to combine information about the (unknown) value of $\lambda(s)$ from different places of the window $[0, n]$. For this reason, the periodicity of λ, that is assumption (1.1), plays a crucial role and leads to the following string of (approximate) equations

$$\begin{aligned}
\lambda(s) &= \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \lambda(s + k\tau) \mathbf{I}\{s + k\tau \in [0, n]\} \\
&\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \int_{[s+k\tau-h_n, s+k\tau+h_n] \cap [0,n]} \lambda(x)dx \\
&= \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \mathbf{E} X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\
&\approx \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]) \\
&\approx \frac{\tau}{n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X([s + k\tau - h_n, s + k\tau + h_n] \cap [0, n]),
\end{aligned} \tag{1.4}$$

where

$$N_n = \#\{k : \ s + k\tau \in [0, n]\}.$$

We note that, in order to make the first $\approx$ in (1.4) works, we require the assumptions that s is a Lebesgue point of λ and (1.2) holds true. We say s is a Lebesgue point of λ, if we have

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{-h}^{h}|\lambda(s+x)-\lambda(s)|dx=0 \tag{1.5}$$

(eg. see [7], p.107-108). Thus, from (1.4) we conclude that the quantity

$$\lambda_n(s):=\frac{\tau}{n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X([s+k\tau-h_n,s+k\tau+h_n]\cap[0,n]), \tag{1.6}$$

can be viewed as an estimator of $\lambda(s)$, provided that the period τ is known. The estimator (1.3) is obtained by replacing τ in (1.6) by $\hat{\tau}_n$.

The idea described in (1.4) and (1.6) of constructing an estimator for $\lambda(s)$ resembles that of [4] where in a similar fashion a non-parametric estimator for an intensity function which, in addition to the periodic trend, also has a polynomial trend. In [4], just like when constructing the estimator $\lambda_n(s)$ in (1.6), the period τ is supposed to be known.

Theorem 1.1. *Let the intensity function λ be periodic and locally integrable. Furthermore, let the bandwidth h_n be such that (1.2) holds true, and*

$$nh_n\rightarrow\infty \tag{1.7}$$

as $n\rightarrow\infty$. If

$$n|\hat{\tau}_n-\tau|/h_n\overset{p}{\rightarrow}0 \tag{1.8}$$

as $n\rightarrow\infty$, then

$$\hat{\lambda}_n(s)\overset{p}{\rightarrow}\lambda(s) \tag{1.9}$$

as $n\rightarrow\infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_n(s)$ is a consistent estimator of $\lambda(s)$.

2. Proofs of Theorem 1.1

Let $B_h(x)$ denotes the interval $[x-h,x+h]$. To establish Theorem 1.1, first we prove

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}X\left(B_{h_n}(s+k\hat{\tau}_n)\cap[0,n]\right)\overset{p}{\rightarrow}\lambda(s), \tag{2.1}$$

as $n\rightarrow\infty$, where $N_n=\#\{k:\ s+k\tau\in[0,n]\}$. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we obtain that the quantity on the l.h.s. of (2.1) is equal to $\lambda(s)+o_p(1)$, as $n\rightarrow\infty$, which of course implies (2.1). Then, to prove (1.9), it remains to check that $\hat{\lambda}_n(s)$ can be replaced by the quantity on the l.h.s. of (2.1), i.e. we must show that the difference between $\hat{\lambda}_n(s)$ and the quantity on the l.h.s. of (2.1) converges in probability to zero, as $n \to \infty$. To show this, first we write this difference as

$$\left(\frac{\hat{\tau}_n N_n}{n} - 1\right) \frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right), \tag{2.2}$$

that is, the quantity on the l.h.s. of (2.1) multiplied by $(\hat{\tau}_n N_n n^{-1} - 1)$. Since $\lambda(s)$ is finite, by (2.1), we have that the quantity on the l.h.s. of (2.1) is $\mathcal{O}_p(1)$, as $n \to \infty$. Hence, it remains to check that

$$\left|\frac{\hat{\tau}_n N_n}{n} - 1\right| = o_p(1), \tag{2.3}$$

as $n \to \infty$. By the triangle inequality, the quantity on the l.h.s. of (2.3) does not exceed

$$\left|\frac{\hat{\tau}_n N_n}{n} - \frac{\hat{\tau}_n}{\tau}\right| + \left|\frac{\hat{\tau}_n}{\tau} - 1\right| \le \frac{\hat{\tau}_n}{n}\left|N_n - \frac{n}{\tau}\right| + \frac{1}{\tau}\left|\hat{\tau}_n - \tau\right|. \tag{2.4}$$

Note that $|n/\tau - N_n| \le 1$, and $\hat{\tau}_n = \mathcal{O}_p(1)$, as $n \to \infty$ (by (1.8)). Hence, the first term on the r.h.s. of (2.4) is $\mathcal{O}_p(n^{-1})$, as $n \to \infty$. By (1.8), we have that its second term is $o_p(n^{-1})$, as $n \to \infty$. Therefore we have (2.3). This completes the proof of Theorem 1.1.

In the following lemma we shall show that we may replace the random centre $s + k\hat{\tau}_n$ of the interval $B_{h_n}(s + k\hat{\tau}_n)$ in (2.1) by its deterministic limit $s + k\tau$.

Lemma 2.1. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.8) are satisfied, then*

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0,n]\right)\right\}\right|$$
$$= o_p(1), \tag{2.5}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: First note that the difference within curly brackets on the l.h.s. of (2.5) does not exceed

$$X\left(B_{h_n}(s + k\hat{\tau}_n) \Delta B_{h_n}(s + k\tau) \cap [0,n]\right). \tag{2.6}$$

Now we notice that

$$B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \subseteq B_{h_n}(s + k\hat{\tau}_n) \subseteq B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau). \tag{2.7}$$

By (2.6) and (2.7) we have

$$\left|\left\{X\left(B_{h_n}(s + k\hat{\tau}_n) \cap [0,n]\right) - X\left(B_{h_n}(s + k\tau) \cap [0,n]\right)\right\}\right|$$
$$\le 2X\left(B_{h_n + |k(\hat{\tau}_n - \tau)|}(s + k\tau) \setminus B_{h_n - |k(\hat{\tau}_n - \tau)|}(s + k\tau) \cap [0,n]\right). \tag{2.8}$$

Hence, to prove (2.5), it suffices to show that

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(B_{h_n+|k(\hat{\tau}_n-\tau)|}(s+k\tau)\setminus B_{h_n-|k(\hat{\tau}_n-\tau)|}(s+k\tau)\cap[0,n]\right)$$
$$= o_p(1), \tag{2.9}$$

as $n\to\infty$. To prove (2.9) we argue as follows. Let Λ_n denotes the l.h.s. of (2.9), and let also $\epsilon>0$ be any fixed real number. Then, for any fixed $\delta>0$, we have that

$$\begin{aligned}\mathbf{P}(|\Lambda_n|\geq\epsilon) &\leq \mathbf{P}(\{|\Lambda_n|\geq\epsilon\}\cap\{n|\hat{\tau}_n-\tau|\leq\delta h_n\})\\ &\quad+\mathbf{P}(n|\hat{\tau}_n-\tau|>\delta h_n).\end{aligned} \tag{2.10}$$

By (1.8), the second term on the r.h.s. of (2.10) is $o(1)$, as $n\to\infty$. While the first term on the r.h.s. of (2.10), does not exceed $\mathbf{P}(|\bar{\Lambda}_n|\geq\epsilon)$, where

$$\bar{\Lambda}_n=\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}X\left(B_{h_n+\delta h_n}(s+k\tau)\setminus B_{h_n-\delta h_n}(s+k\tau)\cap[0,n]\right). \tag{2.11}$$

Next, by Markov inequality for the first moment, we have that

$$\mathbf{P}(|\bar{\Lambda}_n|\geq\epsilon)\leq\epsilon^{-1}\mathbf{E}|\bar{\Lambda}_n|,$$

and $\epsilon^{-1}\mathbf{E}|\bar{\Lambda}_n|$ can also be written as

$$\frac{1}{\epsilon N_n}\sum_{k=-\infty}^{\infty}\frac{1}{h_n}\int_{B_{(1+\delta)h_n}(0)\setminus B_{(1-\delta)h_n}(0)}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx$$
$$=\frac{1}{\epsilon N_n}\frac{1}{h_n}\int_{B_{(1+\delta)h_n}(0)\setminus B_{(1-\delta)h_n}(0)}\lambda(s+x)\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])dx. \tag{2.12}$$

Now we can easily see that

$$\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\leq N_n+1.$$

Then, the r.h.s. of (2.12) does not exceed

$$\frac{1}{\epsilon h_n}\left(\frac{1}{N_n}+1\right)\int_{B_{(1+\delta)h_n}(0)\setminus B_{(1-\delta)h_n}(0)}\lambda(s+x)dx. \tag{2.13}$$

We also can see that $N_n^{-1}+1\leq 2$. Furthermore, the quantity in (2.13) can be bounded above by

$$\frac{2}{\epsilon h_n}\int_{B_{(1+\delta)h_n}(0)}|\lambda(s+x)-\lambda(s)|dx$$
$$+\frac{2}{\epsilon h_n}|B_{(1+\delta)h_n}(0)\setminus B_{(1-\delta)h_n}(0)|\lambda(s). \tag{2.14}$$

Since s is a Lebesgue point of λ, the first term of (2.14) converges to zero as $n\to\infty$. While the second term of (2.14) does not exceed

$8\epsilon^{-1}\delta\lambda(s)$. By taking $\delta = \delta_n \downarrow 0$ as $n \to \infty$, we also have that this term converges to zero as $n \to \infty$. Then we get that $\mathbf{P}(|\Lambda_n| \geq \epsilon) \to 0$ as $n \to \infty$, which is equivalent to (2.9). This completes the proof of Lemma 2.1.

To complete our proof of Theorem 1.1 we also need the following lemma.

Lemma 2.2. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) and (1.7) are satisfied, then*

$$\frac{1}{N_n} \sum_{k=-\infty}^{\infty} \frac{1}{2h_n} \left| X\left(B_{h_n}(s+k\tau) \cap [0,n]\right) - \mathbf{E}X\left(B_{h_n}(s+k\tau) \cap [0,n]\right) \right|$$
$$= o_p(1), \tag{2.15}$$

as $n \to \infty$, provided s is a Lebesgue point of λ.

Proof: First note that, for large n, the random variables

$$X(B_{h_n}(s+k\tau) \cap [0,n]),$$

for all $k \in \mathbb{Z}$, are independent. Then, by Chebyshev's inequality, to prove (2.15) its suffices to check that

$$\left(\frac{1}{2N_n h_n}\right)^2 \sum_{k=-\infty}^{\infty} Var\left\{X\left(B_{h_n}(s+k\tau) \cap [0,n]\right)\right\} = o(1), \tag{2.16}$$

as $n \to \infty$. Since X is a Poisson random variable, $Var(X) = \mathbf{E}X$, and for each k, we can write

$$\mathbf{E}X\left(B_{h_n}(s+k\tau) \cap [0,n]\right)$$
$$= \int_{B_{h_n}(0)} \lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x \in [0,n])dx. \tag{2.17}$$

Because λ is periodic (with period τ), we have that $\lambda(s+k\tau+x) = \lambda(s+x)$, and we also have that $\sum_{k=-\infty}^{\infty} \mathbf{I}(s+k\tau+x \in [0,n]) \leq N_n + 1$. Then, to prove (2.16), its suffices to show

$$\frac{1}{2}\left(\frac{N_n+1}{N_n}\right)\left(\frac{1}{N_n h_n}\right)\left(\frac{1}{2h_n}\int_{B_{h_n}(0)} \lambda(s+x)dx\right) = o(1), \tag{2.18}$$

as $n \to \infty$. Because s is a Lebesgue point of λ, we have

$$(2h_n)^{-1}\int_{B_{h_n}(0)} \lambda(s+x)dx = \lambda(s) + o(1),$$

as $n \to \infty$, which is finite. Because $N_n h_n \to \infty$ as $n \to \infty$, (by (1.7)), then we get (2.18). This completes the proof of Lemma 2.2.

It remains to evaluate a non-random sum.

Lemma 2.3. *Suppose λ is periodic (with period τ) and locally integrable. If, in addition, (1.2) is satisfied, then*

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\mathbf{E}X\left(B_{h_n}(s+k\tau)\cap[0,n]\right)=\lambda(s)+o(1), \qquad (2.19)$$

as $n\to\infty$, provided s is a Lebesgue point of λ.

Proof: Using the fact that X is Poisson, the l.h.s. of (2.19) can be written as

$$\frac{1}{N_n}\sum_{k=-\infty}^{\infty}\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx$$

$$=\frac{1}{2N_nh_n}\int_{-h_n}^{h_n}\lambda(s+x)\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])dx. \qquad (2.20)$$

Now note that

$$(N_n-1)\le\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\le(N_n+1),$$

which implies $N_n^{-1}\sum_{k=-\infty}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])$ can be written as $(1+\mathcal{O}(n^{-1}))$, as $n\to\infty$, uniformly in x. Then, the quantity on the r.h.s. of (2.20) can be written as

$$\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx. \qquad (2.21)$$

By (1.2) together with the assumption that s is a Lebesgue point of λ, we have that $(2h_n)^{-1}\int_{-h_n}^{h_n}\lambda(s+x)dx=\lambda(s)+o(1)$, as $n\to\infty$. Then we get this lemma. This completes the proof of Lemma 2.3.

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