WEAK AND STRONG CONVERGENCE OF A KERNEL-TYPE ESTIMATOR FOR THE INTENSITY OF A PERIODIC POISSON PROCESS

I WAYAN MANGKU

Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Bogor Agricultural University
Jl. Meranti, Kampus IPB Darmaga, Bogor, 16680 Indonesia

Abstract. In this paper we survey some results on weak and strong convergence of kernel type estimators for the intensity of a periodic Poisson process. We consider the situation when the period is known in order to be able to present simple proofs of the results. For the more general results, which includes the case when the period is unknown, we refer to [15], [16].

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1. Introduction

In this paper we consider kernel type estimation of the intensity function λ at a given point $s \in [0, n]$, using only a single realization $N(\omega)$ of the periodic Poisson process N observed in $[0, n]$. This problem arises frequently in many diverse areas including:

- Communications (cf., e.g., [23], [24], [17], [13], [2])
- Hydrology, Meteorology (cf., e.g., [12], [34], [38], [39], [40], [1], [32], [21], [14], [9], [10])
- Insurance, Reliability (cf., e.g., [3], [11])
- Medical Sciences (cf., e.g., [30], [31] [22], [5], [33])
- Seismology (cf., e.g., [35], [36], [37], [26], [27], [28], [25]).

Some of these can also be found in the monographs by [4], [8], [18], [7], [6], [19], [29], [20], and others.

Let N be a Poisson process on $[0, \infty)$ with (unknown) locally integrable intensity function λ. We assume that λ is a periodic function with (known) period τ. We do not assume any parametric form of λ, except that it is periodic. That is, for each point $s \in [0, \infty)$ and all

$k \in \mathbf{Z}$, with $\mathbf{Z}$ denotes the set of integers, we have

$$\lambda(s + k\tau) = \lambda(s). \tag{1.1}$$

Suppose that, for some $\omega \in \Omega$, a single realization $N(\omega)$ of the Poisson process N defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with intensity function λ is observed, though only within a bounded interval $[0, n]$. Our goal in this paper is: (a) To study construction of a kernel-type estimator for λ at a given point $s \in [0, \infty)$ using only a single realization $N(\omega)$ of the Poisson process N observed in interval $[0, n]$. (b) To study the minimal conditions for having weak and strong convergence of this estimator.

We will assume throughout that s is a Lebesgue point of λ, that is we have

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{-h}^{h} |\lambda(s + x) - \lambda(s)| dx = 0 \tag{1.2}$$

(eg. see [41], p.107-108).

Note that, since λ is a periodic function with period τ, the problem of estimating λ at a given point $s \in [0, \infty)$ can be reduced into a problem of estimating λ at a given point $s \in [0, \tau)$. Hence, for the rest of this paper, we will assume that $s \in [0, \tau)$.

Note also that, the meaning of the asymptotic $n \to \infty$ in this paper is somewhat different from the classical one. Here n does not denote our sample size, but it denotes the length of the interval of observations. The size of our samples is a random variable denoted by $N([0, n])$.

2. Construction of the estimator and results

Let $K : \mathbf{R} \to \mathbf{R}$ be a real valued function, called *kernel*, which satisfies the following conditions: (K1) K is a probability density function, (K2) K is bounded, and (K3) K has (closed) support $[-1, 1]$. Let also h_n be a sequence of positive real numbers converging to 0, that is,

$$h_n \downarrow 0, \tag{2.1}$$

as $n \to \infty$.

Using the introduced notations, we may define the estimator of λ at a given point $s \in [0, \tau)$ as follows

$$\hat{\lambda}_{n,K}(s) := \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x - (s + k\tau)}{h_n}\right) N(dx). \tag{2.2}$$

For a more general kernel-type estimator of the intensity of a periodic Poisson process, which includes the case when the period τ has to be estimated, we refer to Helmers, Mangku and Zitikis ([15], [16]).

Next we describe the idea behind the construction of the kernel-type estimator $\hat{\lambda}_{n,K}(s)$ of $\lambda(s)$. First, note that since there is available only one realization of the Poisson process N, we have to collect necessary information about the (unknown) value of $\lambda(s)$ from different places of the interval $[0, n]$. For this reason, assumption (1.1) plays a crucial role and leads to the following string of (approximate) equations. Let

$$N_n = \#\{k : \ s + k\tau \in [0, n]\},$$

where # denotes the number of elements. Then we have

$$\begin{aligned}
\lambda(s) &= \frac{1}{N_n}\sum_{k=0}^{\infty} \lambda(s+k\tau)\mathbf{I}\{s+k\tau \in [0,n]\} \\
&\approx \frac{1}{N_n}\sum_{k=0}^{\infty} \frac{1}{2h_n}\int_{[s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]} \lambda(x)dx \\
&= \frac{1}{N_n}\sum_{k=0}^{\infty} \frac{1}{2h_n}\mu([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]) \\
&\approx \frac{1}{N_n}\sum_{k=0}^{\infty} \frac{1}{2h_n}N([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]) \\
&\approx \frac{\tau}{n}\sum_{k=0}^{\infty} \frac{1}{2h_n}N([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]), \quad (2.3)
\end{aligned}$$

where $\mathbf{I}$ denotes the indicator function and μ denotes the measure defined as

$$\mu(A) := \mathbf{E}N(A) = \int_A \lambda(x)dx, \quad A \in \mathcal{B}(\mathbf{R}).$$

We note that in order to make the first $\approx$ in (2.3) work, we have assumed that s is a Lebesgue point of λ and h_n converges to 0. Thus, from (2.3) we conclude that

$$\hat{\lambda}_n(s) := \frac{\tau}{n}\sum_{k=0}^{\infty} \frac{1}{2h_n}N([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]), \qquad (2.4)$$

is an estimator of $\lambda(s)$. Note that the estimator $\hat{\lambda}_n(s)$ can be rewritten as

$$\hat{\lambda}_n(s) = \frac{\tau}{n}\sum_{k=0}^{\infty} \frac{1}{h_n}\int_0^n \frac{1}{2}\mathbf{I}_{[-1,1]}\left([s+k\tau-h_n, s+k\tau+h_n]\right)N(dx). \qquad (2.5)$$

By replacing the function $\frac{1}{2}\mathbf{I}_{[-1,1]}(\cdot)$ in (2.5) by the general kernel K, we immediately arrive at the estimator introduced in (2.2).

Theorem 2.1. (Weak Convergence)

Suppose that the intensity function λ is periodic and locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, the bandwidth h_n satisfies assumptions (2.1) and

$$n\ h_n \to \infty, \qquad (2.6)$$

then

$$\hat{\lambda}_{n,K}(s) \xrightarrow{p} \lambda(s), \tag{2.7}$$

as $n \to \infty$, *provided* s *is a Lebesgue point of* λ. *In other words,* $\hat{\lambda}_{n,K}(s)$ *is a consistent estimator of* $\lambda(s)$. *In addition, the Mean-Squared-Error (MSE) of* $\hat{\lambda}_{n,K}(s)$ *converges to* 0, *as* $n \to \infty$, *that is we have*

$$MSE(\hat{\lambda}_{n,K}(s)) \;\to\; 0, \tag{2.8}$$

as $n \to \infty$, *provided* s *is a Lebesgue point of* λ.

Theorem 2.2. (Strong Convergence)

Suppose that the intensity function λ *is periodic and locally integrable. If the kernel* K *satisfies conditions* $(K1), (K2), (K3)$, *the bandwidth* h_n *satisfies assumptions (2.1) and*

$$\sum_{n=1}^{\infty} \exp\{-\epsilon\sqrt{nh_n}\} < \infty, \tag{2.9}$$

for each $\epsilon > 0$, *then*

$$\hat{\lambda}_{n,K}(s) \xrightarrow{a.s.} \lambda(s), \tag{2.10}$$

as $n \to \infty$, *provided* s *is a Lebesgue point of* λ. *In other words,* $\hat{\lambda}_{n,K}(s)$ *is a strong consistent estimator of* $\lambda(s)$.

3. Proofs of Theorem 2.1

To prove Theorem 2.1, we need the following two lemmas.

Lemma 3.1. (Asymptotic unbiasedness)

Suppose that the intensity function λ *is periodic and locally integrable. If the kernel* K *satisfies conditions* $(K1), (K2), (K3)$, *and* h_n *satisfies assumptions (2.1), then*

$$\mathbf{E}\hat{\lambda}_{n,K}(s) \;\to\; \lambda(s), \tag{3.1}$$

as $n \to \infty$, *provided* s *is a Lebesgue point of* λ.

Lemma 3.2. (Convergence of the variance)

Suppose that the intensity function λ *is periodic and locally integrable. If the kernel* K *satisfies conditions* $(K1), (K2), (K3)$, *and* h_n *satisfies assumptions (2.1) and (2.6), then*

$$Var\left(\hat{\lambda}_{n,K}(s)\right) \;\to\; 0, \tag{3.2}$$

as $n \to \infty$, *provided* s *is a Lebesgue point of* λ.

Proof of Lemma 3.1

Note that

$$
\begin{aligned}
\mathbf{E}\hat{\lambda}_{n,K}(s) &= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\mathbf{E}N(dx) \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)dx \\
&= \frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x-(s+k\tau)}{h_n}\right)\lambda(x)\mathbf{I}(x\in[0,n])dx.
\end{aligned}
\tag{3.3}
$$

By a change of variable and using (1.1), we can write the r.h.s. of (3.3) as

$$
\begin{aligned}
&\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
= &\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\lambda(x+s)\mathbf{I}(x+s+k\tau\in[0,n])dx.
\end{aligned}
\tag{3.4}
$$

We will prove this lemma, by showing that the quantity on the r.h.s. of (3.4) is equal to $\lambda(s)+o(1)$, as $n\to\infty$. To check this, note that the r.h.s. of (3.4) can be written as

$$
\begin{aligned}
&\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)(\lambda(x+s)-\lambda(s))\,\mathbf{I}(x+s+k\tau\in[0,n])dx \\
&+\frac{\lambda(s)\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
= &\frac{\tau}{n\ h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)(\lambda(x+s)-\lambda(s))\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])dx \\
&+\frac{\lambda(s)\tau}{n\ h_n}\int_{\mathbf{R}} K\left(\frac{x}{h_n}\right)\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])dx.
\end{aligned}
\tag{3.5}
$$

Now note that

$$
\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n]) = \frac{n}{\tau}+\mathcal{O}(1),
\tag{3.6}
$$

as $n \to \infty$ uniformly in $x \in [-h_n, h_n]$. Then, the r.h.s. of (3.5) can be written as

$$\begin{aligned} & \frac{\tau}{n\ h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) (\lambda(x+s) - \lambda(s)) \left(\frac{n}{\tau} + \mathcal{O}(1)\right) dx \\ & + \frac{\lambda(s)\tau}{n\ h_n} \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \left(\frac{n}{\tau} + \mathcal{O}(1)\right) dx \\ = & \int_{\mathbf{R}} K\left(\frac{x}{h_n}\right) \frac{1}{h_n} (\lambda(x+s) - \lambda(s))\, dx + \lambda(s) \int_{\mathbf{R}} K(x) dx \\ & + \mathcal{O}\left(\frac{1}{n}\right), \end{aligned} \tag{3.7}$$

as $n \to \infty$. Since s is a Lebesque of λ (cf. (1.2)) and the kernel K satisfies conditions (K2) and (K3), it is easily seen that the first term on the r.h.s. of (3.7) is $o(1)$, as $n \to \infty$. By the assumption: $\int_{\mathbf{R}} K(x)dx = 1$ (cf. (K1)), the second term on the r.h.s. of (3.7) is equal to $\lambda(s)$. Clearly, the third term on the r.h.s. of (3.7) is $o(1)$, as $n \to \infty$. Hence, the r.h.s. of (3.4) is equal to $\lambda(s) + o(1)$, as $n \to \infty$. This completes the proof of Lemma 3.1.

Proof of Lemma 3.2

The variance of $\hat{\lambda}_{n,K}(s)$ can be computed as follows

$$Var\left(\hat{\lambda}_{n,K}(s)\right) = \frac{\tau^2}{n^2} Var\left(\sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx)\right). \tag{3.8}$$

By (2.1), for sufficiently large n, we have that the intervals $[s+k\tau - h_n, s+k\tau+h_n]$ and $[s+j\tau-h_n, s+j\tau+h_n]$ are not overlap for all $k \neq j$. This implies, for all $k \neq j$,

$$K\left(\frac{x-(s+k\tau)}{h_n}\right) N(dx) \text{ and } K\left(\frac{x-(s+j\tau)}{h_n}\right) N(dx)$$

are independent. Hence, the r.h.s. of (3.8) can be computed as follows

$$\begin{aligned} & \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) Var(N(dx)) \\ = & \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) \mathbf{E}N(dx) \\ = & \frac{\tau^2}{n^2 h_n^2} \sum_{k=0}^{\infty} \int_0^n K^2\left(\frac{x-(s+k\tau)}{h_n}\right) \lambda(x)dx. \end{aligned} \tag{3.9}$$

By a change of variable and using (1.1), the r.h.s. of (3.9) can be written as

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2}\sum_{k=0}^{\infty}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\lambda(x+s+k\tau)\mathbf{I}(x+s+k\tau\in[0,n])dx \\
=\ &\frac{\tau^2}{n^2h_n^2}\sum_{k=0}^{\infty}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\lambda(x+s)\mathbf{I}(x+s+k\tau\in[0,n])dx. \quad (3.10)
\end{aligned}$$

The r.h.s. of (3.10) is equal to

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(\lambda(x+s)-\lambda(s))\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])dx \\
&+\frac{\lambda(s)\tau^2}{n^2h_n^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\sum_{k=0}^{\infty}\mathbf{I}(x+s+k\tau\in[0,n])dx. \quad (3.11)
\end{aligned}$$

By (3.6), the quantity in (3.11) can be written as

$$\begin{aligned}
&\frac{\tau^2}{n^2h_n^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)(\lambda(x+s)-\lambda(s))\left(\frac{n}{\tau}+\mathcal{O}(1)\right)dx \\
&+\frac{\lambda(s)\tau^2}{n^2h_n^2}\int_{\mathbf{R}} K^2\left(\frac{x}{h_n}\right)\left(\frac{n}{\tau}+\mathcal{O}(1)\right)dx. \quad (3.12)
\end{aligned}$$

Since the kernel K is bounded and has support in $[-1,1]$, by (1.2), we see that the first term on the r.h.s. of (3.12) is of order $o(n^{-1}(h_n)^{-1}))$, as $n\to\infty$. By the assumption (2.6), we have that this term is $o(1)$, as $n\to\infty$. A simple argument shows that the second term on the r.h.s. of (3.12) is of order $\mathcal{O}(n^{-1}(h_n)^{-1}))=o(1)$, as $n\to\infty$. This completes the proof of Lemma 3.2.

Proof of Theorem 2.1

By Lemma 3.1 and Lemma 3.2 we directly obtain (2.8). To prove (2.7), we have to show, for each $\epsilon>0$,

$$\mathbf{P}\left(|\hat{\lambda}_{n,K}(s)-\lambda(s)|>\epsilon\right)\ \rightarrow\ 0, \quad (3.13)$$

as $n\to\infty$. To prove (3.13), we argue as follows. By Lemma 3.1, there exist a large constant n_0 such that $|\mathbf{E}\hat{\lambda}_{n,K}(s)-\lambda(s)|\leq 1/2$, for all $n>n_0$. Hence, for sufficiently large n, the probability on the l.h.s. of (3.13) does not exceed

$$\mathbf{P}\left(|\hat{\lambda}_{n,K}(s)-\mathbf{E}\hat{\lambda}_{n,K}(s)|>\frac{\epsilon}{2}\right)\leq\frac{4Var(\hat{\lambda}_{n,K}(s))}{\epsilon^2}, \quad (3.14)$$

by the Chebyshev inequality. By Lemma 3.2, we have the r.h.s. of (3.14) converges to 0, as $n\to\infty$. Hence we obtain (3.13). This completes the proof of Theorem 2.1.

4. Proofs of Theorem 2.2

By the Borel-Cantelli lemma, to verify Theorem 2.2, it suffices to prove the following theorem.

Theorem 4.1. (Complete Convergence)

Suppose that the intensity function λ is periodic and locally integrable. If the kernel K satisfies conditions $(K1), (K2), (K3)$, the bandwidth h_n satisfies assumptions (2.1) and (2.9), then

$$\hat{\lambda}_{n,K}(s) \ \stackrel{c}{\rightarrow} \ \lambda(s), \tag{4.1}$$

as $n \to \infty$, provided s is a Lebesgue point of λ. In other words, $\hat{\lambda}_{n,K}(s)$ converges completely to $\lambda(s)$, as $n \to \infty$.

Proof: To prove (4.1), we have to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\hat{\lambda}_{n,K}(s) - \lambda(s)| > \epsilon \right) < \infty. \tag{4.2}$$

Since the probability on the l.h.s. of (3.13) does not exceed the probability on the l.h.s. of (3.14), to prove (4.2), it suffices to show, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|\hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)| > \frac{\epsilon}{2} \right) < \infty. \tag{4.3}$$

Let $D_n = \hat{\lambda}_{n,K}(s) - \mathbf{E}\hat{\lambda}_{n,K}(s)$, that is

$$\begin{aligned} D_n \ := \ & \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n} \right) N(dx) \\ & - \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \int_0^n K\left(\frac{x-(s+k\tau)}{h_n} \right) \lambda(x) dx. \end{aligned}$$

Then, to prove (4.3), it suffices to show that, for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\left(|D_n| > \epsilon \right) < \infty. \tag{4.4}$$

To prove (4.4), we argue as follows. For every $t > 0$, we have that

$$\mathbf{P}\left(|D_n| \geq c_1 \epsilon \right) \leq \exp\{-c_1 \epsilon t\} \big(\mathbf{E} \exp\{t D_n\} + \mathbf{E} \exp\{-t D_n\} \big). \tag{4.5}$$

To make our further considerations more transparent, we denote

$$Y_k := \int_0^n K\left(\frac{x-(s+k\tau)}{h_n} \right) N(dx)$$

and then rewrite D_n as

$$D_n = \frac{\tau}{n} \sum_{k=0}^{\infty} \frac{1}{h_n} \{ Y_k - \mathbf{E} Y_k \}. \tag{4.6}$$

Since $h_n \downarrow 0$, the random variables Y_k, $k = 1, 2, ...$ are independent for all sufficiently large n (depending on the period τ). Thus, for sufficiently large n, we obtain

$$\mathbf{E}\exp\{\pm t D_n\} = \prod_{k=0}^{\infty} \mathbf{E}\exp\left\{\pm \frac{t\tau}{nh_n}(Y_k - \mathbf{E}Y_k)\right\}. \tag{4.7}$$

Using the well known formula for the Laplace transform of the Poisson process, we obtain that

$$\mathbf{E}\exp\left\{\pm\frac{t\tau}{nh_n}Y_k\right\} = \exp\left\{\int_0^n (e^{K^*(x)} - 1)\lambda(x)dx\right\}, \tag{4.8}$$

where we used the notation

$$K^*(x) := \pm\frac{t\tau}{nh_n}K\left(\frac{x-(s+k\tau)}{h_n}\right).$$

Consequently, for every factor on the r.h.s. of (4.7) we have the following formula

$$\begin{aligned}&\mathbf{E}\exp\left\{\pm\frac{t\tau}{nh_n}\{Y_k - \mathbf{E}Y_k\}\right\} \\ &= \exp\left\{\int_0^n (e^{K^*(x)} - 1 - K^*(x))\lambda(x)dx\right\}.\end{aligned} \tag{4.9}$$

Since $|\exp(x) - 1 - x|$ does not exceed $x^2\exp(|x|)$, we obtain from (4.9) that

$$\begin{aligned}&\mathbf{E}\exp\left\{\pm\frac{t\tau}{nh_n}\{Y_k - \mathbf{E}Y_k\}\right\} \\ &\le \exp\left\{\int_0^n |K^*(x)|^2 e^{|K^*(x)|}\lambda(x)dx\right\}.\end{aligned} \tag{4.10}$$

We now make the following choice

$$t := \frac{1}{c_1}\sqrt{\frac{nh_n}{\tau}}. \tag{4.11}$$

Using the assumption that K is bounded and has support in the interval $[-1, 1]$, we obtain from (4.10) with (4.11) that

$$\begin{aligned}&\mathbf{E}\exp\left\{\pm\frac{t\tau}{nh_n}\{Y_k - \mathbf{E}Y_k\}\right\} \\ &\le \exp\left\{c\frac{\tau}{nh_n}\mu\left([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]\right)\right\},\end{aligned} \tag{4.12}$$

for a constant c that does not depend on n. Applying bound (4.12) on the r.h.s. of (4.7), we obtain

$$\begin{aligned} &\mathbf{E}\exp\{\pm tD_n\} \\ &\leq \exp\left\{c\frac{\tau}{n}\sum_{k=0}^{\infty}\frac{1}{h_n}\mu\left([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]\right)\right\}. \end{aligned} \tag{4.13}$$

Furthermore, we note that the quantity $\mu\left([s+k\tau-h_n, s+k\tau+h_n]\cap[0,n]\right)$ obviously equals to

$$\int_{-h_n}^{h_n}\lambda(s+k\tau+x)\mathbf{I}(s+k\tau+x\in[0,n])dx.$$

Consequently, using the periodicity of λ and the fact that

$$\sum_{k=0}^{\infty}\mathbf{I}(s+k\tau+x\in[0,n])\in\left[\frac{n}{\tau}-1,\frac{n}{\tau}+1\right]$$

on the r.h.s. of (4.13), we obtain that

$$\mathbf{E}\exp\{\pm tD_n\}\leq\exp\left\{c\frac{1}{h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx\right\}.$$

Since s is a Lebesgue point of λ, we have that

$$\frac{1}{2h_n}\int_{-h_n}^{h_n}\lambda(s+x)dx\to\lambda(s),$$

when $n\to\infty$. Thus,

$$\lim_{n\to\infty}\mathbf{E}\exp\{\pm tD_n\}\leq c<\infty. \tag{4.14}$$

Bound (4.14), when applied on the r.h.s. of (4.5), implies that

$$\mathbf{P}\left(|D_n|\geq\epsilon\right)\leq\exp\left\{-\epsilon\sqrt{\frac{n}{\tau}h_n}\right\}=\exp\left\{-\epsilon^*\sqrt{nh_n}\right\},$$

due to our choice of t as in (4.11). By the assumption (2.9), we obtain (4.1). This completes the proof of Theorem 4.1.

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