ON THE STRUCTURE OF QUANTIZABLE ALGEBRAS OF THE PRODUCTS OF SYMPLECTIC MANIFOLDS WITH POLARIZATIONS

M.F. Rosyid[1]

[1]Working Group on Mathematical Physics, Dept of Physics, UGM, Sekip unit III, Yogyakarta 55281
and
Institute for Science in Yogyakarta (I-Es-Ye) Yogyakarta

ABSTRACT

A canonical polarization of symplectic product of two symplectic manifolds with polarization is constructed from the polarizations of each symplectic manifold. The polarizations is referred to as product polarization. The structure of quantizable algebras of symplectic products of two symplectic manifolds with respect to the product polarization in the sense of the geometric quantization is studied. The so-called GQ-consistent kinematical algebras of the products are extracted from the quantizable algebras.

Keywords : Quantization, Differential Geometry

Makalah diterima 3 Agustus 2007

1. INTRODUCTION

Physics is the attempt to find out the patterns of natural phenomenon. The patterns under question are modeled by suitable mathematical objects and the relations among them. Therefore, physics can be understood also as the attempt to *pick* or to *construct* (new) mathematical models suitable for describing the patterns of natural phenomenon. Thus, physics is an attempt to represent physical realities in mathematical realities. In somewhat provocative words, it could be said that physics is a branch of mathematics in which experimental data play an essential role as constraints.

The suitable and rigorously mathematical model for classical mechanics[1] is *symplectic geometry* (Libermann, and Marle, 1987). On the other side, Hilbert spaces and operator algebras in the spaces are appropriate mathematical models for quantum mechanics (Emch, 1984). Then, with a quantization we mean an attempt to make the classical description of natural phenomenon accessible from the quantum mechanical one, or vice versa.

The present work is situated in the tradition of (Rosyid, 2003; Rosyid, 2005a; Rosyid, 2005b, Rosyid, 2005c; Rosyid, 2007a; Rosyid, 2007b), namely in the "junction" area between geometric and Borel quantization. A canonical polarization of symplectic product of two symplectic manifolds with reducible polarization is constructed from the polarizations of each symplectic manifold. The

[1] Classical Mechanics is proposed as the pattern of macroscopic phenomenon while quantum mechanics is understood as the pattern of microscopic ones.

polarization is referred to as product polarization. The structure of quantizable algebras of symplectic products of two symplectic manifolds with respect to the product polarization in the sense of the geometric quantization is studied. The structure of the so called GQ-consistent kinematical algebra of symplectic product of symplectic manifolds is investigated. The above mentioned scheme can be applied for instance to a physical system consisting of a massive particle with spin.

2. GQ-CONSISTENT KINEMATICAL ALGEBRA

Let (M,w) be a quantizable symplectic manifold of dimension $2n$ and P a strong integrable complex polarization so that $0 \leq \dim(D) \leq n$, where $D^C = P \cap P^*$ is the isotropic distribution associated with P(see Rosyid, 2003). In the -(1/2+iγ)-P-densities quantization, a function $f \in C^\infty(M,R)$ is said to be quatizable if the Hamiltonian vector field X_f generated by f preserves the polarization P in the sense[2] of $[X_f,P] \subset P$. Let $F_P(M,R)$ be the set of all real quantizable functions and

$$F(M,R;D) := \{f \in F_P(M,R) \mid [X_f, D] \subset D\}$$

as the set of all quantizable functions preserving the distribution D.

Let Z be a vector field on M. The maping $\pi_{D*}Z$ defined by $\pi_{D*}Z(\pi_D(m)) = \pi_{D*}|_m Z$ determines a smooth vector field on M/D if and only if Z preserves the distribution D, i.e. $[Z,D] \subset D$ (see Rosyid, 2005c). It is clear therefore that a quantizable function f belongs to $F(M, R; D)$ if and only if there exists a differentiable vector field X on M/D, so that X_f and X are π_D-related, i.e. $\pi_{D*}X_f$ is a differentiable vector field on M/D.

The set $X(M;D)$ of all vector fields on M preserving the distribution D forms a Lie subalgebra of $X(M)$, the set of all smooth vector fields on M. Now let $X_F(M;D)$ denote the set of all Hamiltonian vector fields on M generated by the functions in the set $F(M,R;D)$ and p^F_{D*} be defined as the restriction of the Lie homomorphism $p_{D*} : X(M;D) \to X(M/D)$ to $X_F(M;D)$. The set $X_F(M;D)$ is clearly a Lie subalgebra of $X(M;D)$ and p^F_{D*} is a Lie homomophism.

Let $X_F^{\sim}(M;D)$ denote the quotient algebra of $X_F(M;D)$ relative to the kernel $K(p^F_{D*})$ of p^F_{D*} and $[.,.]^{\sim}$ be the quotient bracket in $X_F^{\sim}(M;D)$. If $X_F{}^{\sim}{}_c(M;D)$ is the subset of $X_F^{\sim}(M;D)$ defined by

$$X_F{}^{\sim}{}_c(M;D) = \{[X_f] \in X_F^{\sim}(M;D) \mid p_{D*}[X_f] \in X_c(M/D)\},$$

where $X_c(M/D)$ is the set of all complete vector fields on M/D, then, in general, the identity $p_{D*}X_F^{\sim}(M;D) = X_c(M/D)$, is not respected, where $p_{D*}[X_f]$ is defined as $p_{D*}X_{f'}$ for arbitrary $X_{f'} \in [X_f]$. Furthermore, the maping

$$p^c_{D*} : X_F{}^{\sim}{}_c(M;D) \to X_c(M/D)$$

which is defined by $p^c_{D*}[X_f] = p_{D*}X_{f'}$ for arbitrary $X_{f'} \in [X_f]$ is an injective partial Lie homomorphism.

Define now X as the assignment $X_f \in X_F(M;D)$ to f in $F(M,R;D)$ and let $K(p^F_{D*})$ denote the kernel of the Lie homomorphism p^F_{D*} in the set $X_F(M;D)$. Further, let $F_K(M,R;D)$ be the inverse image of $K(p^F_{D*})$ under the maping X. If g is contained in $F_K(M,R;D)$, then $p^F_{D*}X_g$ vanishes. Therefore, there exists uniquely a real smooth function $\zeta_g \in C^\infty(M/D,R)$ so that $g = \zeta_g ? \pi_D$. However, the converse is in

---

[2] The expression $[X_f, P] \subset P$ means that $[X_f,Y]$ is a cross section of P for every cross section Y of P. A cross section Y of P is a vector field on M so that $Y(m) \in P(m)$ for every $m \in M$.

general not correct. If $C^\infty_D(M,R)$ is the set of all real smooth functions on M which are constant on every leaf of D and P is real (so that $D = \bar{D}$), then $F_K(M,R;D) = C^\infty_D(M,R)$. Since, $C^\infty_D(M,R)$ is equal to

$$p_D^* C^\infty(M/D,R) := \{\zeta \circ \pi_D \in C^\infty(M,R) \mid \zeta \in C^\infty(M/D,R)\},$$

it follows also that

$$F_K(M,R;D) = p_D^* C^\infty(M/D,R).$$

Now let $\Xi : F_K(M,R;D) \to C^\infty(M/D,R)$ be the injection defined by $\Xi(g) = \zeta_g$ and define for every equivalent class $[X_f] \in X_{\tilde{F}c}(M;D)$ a linear operator $L_{[X_f]}$ on the algebra $F_K(M,R;D)$ so that

$$L_{[X_f]} = X_{f'}(g),$$

for every $g \in F_K(M,R;D)$ and arbitrary $X_{f'} \in [X_f]$.

Let $S_{GQ}(M;D)$ be the semidirect sum $F_K(M,R;D) \oplus_s X_{\tilde{F}c}(M;D)$ with Lie bracket $[\cdot,\cdot]^s$ defined by

$$[(g_1,[X_f]), (g_2,[X_{f'}])]^s = (L_{[X_f]} g_2 - L_{[X_{f'}]} g_1, [[X_f],[X_{f'}]]^\sim),$$

for all $[X_f]$, $[X_{f'}] \in X_{\tilde{F}c}(M;D)$ with $[[X_f],[X_{f'}]]^\sim \in X_{\tilde{F}c}(M;D)$ and all g_1, $g_2 \in F_K(M,R;D)$. The pair $(S_{GQ}(M;D), [\cdot,\cdot]^s)$ is called *GQ-consistent kinematical algebra* in (M,w,P).

Define a maping, denoted by $\Xi \oplus_s p^c_{D*}$, from the algebra $S_{GQ}(M;D)$ into the kinematical algebra $S(M/D) = C^\infty(M/D,R) \oplus_s X_c(M/D)$ through

$$\Xi \oplus_s p^c_{D*}(g, [X_f]) = (\Xi(g), p^c_{D*}[X_f]),$$

for every $(g, [X_f]) \in S_{GQ}(M;D)$. The maping $\Xi \oplus_s p^c_{D*}$ is clearly a partial Lie homomorphism. The GQ-consistent kinematical algebra $S_{GQ}(M;D)$ is said to be *almost complete* whenever the maping $\Xi \oplus_s p^c_{D*}$ is a partial Lie isomorphism. Furthermore, $S_{GQ}(M;D)$ is said to be *complete* if it is almost complete and $X_{\tilde{F}c}(M;D)$ is equal to $X_{\tilde{F}}(M;D)$.

The ideal $F_K(M,R;D)$ in the algebra $S_{GQ}(M;D)$ is associated with the localization of the physical system or particle in its configuration space. Then, the elements of $F_K(M,R;D)$ represent position of the physical system.

Symplectic Product

Let (M_1,w_1) and (M_2,w_2) be two symplectic manifolds of dimension $2n_1$ and $2n_2$ respectively. Furthermore, let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the canonical projections.

Definition 1 : *The symplectic manifold (M,w), where $M = M_1 \times M_2$ and $w = p_1^* w_1 + p_2^* w_2$ is called* ***symplectic product*** *of (M_1,w_1) and (M_2,w_2).*

Let P_1 be a reducible polarization on (M_1,w_1) and P_2 a reducible polarization on (M_2,w_2). Let $P_1 \times P_2$ denote the distribution defined by

$$P_1 \times P_2 |_{(m_1,m_2)} = P_1 |_{m_1} \oplus P_2 |_{m_2},$$

for every $(m_1, m_2) \in M$.

Proposition 1 : *The distribution $P_1 \times P_2$ is a reducible polarization of $M = M_1 \times M_2$. The polarization is referred to as* ***product polarization***.

Now we mention an example of physical systems which admits the above structure. Let G denote the Poincaré group, i.e. the symmetry group of relativistic elementary particles and $\mathcal{G}$ the Lie algebra of G. The algebra $\mathcal{G}$ is spanned by the generators $\{P_\nu ; \nu = 0,1,2,3\}$ of translations and $\{J_{m\nu} ; \mu,\nu = 0,1,2,3\}$

of Lorentz transformations. Define now P^2 and w^2 as $P^2 = h^{mv} P_\mu P_v$ and $w^2 = h_{mv} w^\mu w^v$, where $w^\rho = (1/2)\varepsilon^{r\mu vt} J_{\mu v} P_\tau$ and h^{mv} is the Lorentz metric. The orbit of G in $\mathcal{G}^*$ which plays the role of classical phase space of relativistic elementary particles is characterized by P^2 and w^2 (see Landsman, 1998; Simms and Woodhouse, 1976). If $P^2 > 0$ and w^2 does not vanish, i.e. if the particles under consideration are massive, the orbit of G in $\mathcal{G}^*$ is diffeomorphic to $R^6 \times S^2$. The canonical symplectic structure w_G on each orbit is given by Def.1, i.e. $w_G = p_R^* w_R + p_S^* w_S$, where w_R and w_S is the canonical symplectic structure of R^6 and S^2, respectively.

3. GQ-CONSISTENT KINEMATICAL ALGEBRA IN SYMPLECTIC PRODUCT

Let $\mathcal{X}(M;P)$ and $\mathcal{X}(M_i;P_i)$ (i = 1, 2) denote the set of all vector fields on M and M_i which preserve P and P_i, respectively. Furthermore, let $D = P \cap P^* \cap TM$, $D_1 = P_1 \cap P_1^* \cap TM_1$, and $D_2 = P_2 \cap P_2^* \cap TM_2$. Finally, let $\mathcal{X}(M;D)$ and $\mathcal{X}(M_i;D_i)$ (i = 1, 2) denote the set of all vector fields on M and M_i which preserve D and D_i, respectively.

Proposition 2 : *The cartesian product* $\mathcal{X}(M_1;P_1) \times \mathcal{X}(M_2;P_2)$ *is contained in the set* $\mathcal{X}(M;P)$ *and* $\mathcal{X}(M_1;D_1) \times \mathcal{X}(M_2;D_2)$ *contained in* $\mathcal{X}(M;D)$

Let $\mathcal{X}(M;D_1)$ be the set of all vector fields on M of the form $(X_1,0)$ with $X_1 \in \mathcal{X}(M_1;D_1)$ and $\mathcal{X}(M;D_2)$ the set of all vector fields on M of the form $(0, X_2)$ with $X_2 \in \mathcal{X}(M_2;D_2)$.

Remark 1 : $\mathcal{X}(M;D_1)$ *and* $\mathcal{X}(M;D_2)$ *are Lie subalgebra of* $\mathcal{X}(M;D)$. *Moreover, the algebra* $\mathcal{X}(M_1;D_1) \times \mathcal{X}(M_2;D_2)$ *is equal to* $\mathcal{X}(M;D_1) \oplus \mathcal{X}(M;D_2)$.

Let $\mathcal{F}_P(M,R)$ denote the set of all quantizable functions on (M,ω) relative to the polarization $P = P_1 \times P_2$ and $\mathcal{F}_{P_i}(M_i,R)$ denote the set of all quantizable functions on (M_i,w_i) relative to the polarization P_i (i = 1,2). Now define $\mathcal{X}_\mathcal{F}(M;P)$ as the set of all Hamiltonian vector fields generated by all functions contained in $\mathcal{F}_P(M,R)$ and $\mathcal{X}_\mathcal{F}(M_i;P_i)$ as the set of all Hamiltonian vector fields generated by all functions contained in $\mathcal{F}_{P_i}(M_i,R)$.

Proposition 3 : *The set* $\mathcal{X}_\mathcal{F}(M_1;P_1) \times \mathcal{X}_\mathcal{F}(M_2;P_2)$ *is contained in* $\mathcal{X}_\mathcal{F}(M;P)$. *Every function* $f \in \mathcal{F}_P(M,R)$ *so that* $X_f \in \mathcal{X}_\mathcal{F}(M_1;P_1) \times \mathcal{X}_\mathcal{F}(M_2;P_2)$ *can be written as the sum* $f = f_1 ? \pi_1 + f_2 ? \pi_2$, *where* $f_1 \in \mathcal{F}_{P_1}(M_1,R)$ *and* $f_1 \in \mathcal{F}_{P_2}(M_2,R)$.

Next, let $\mathcal{X}_\mathcal{F}(M;D)$ and $\mathcal{X}_\mathcal{F}(M_i;D_i)$ (i = 1,2) be the set defined by

$$\mathcal{X}_\mathcal{F}(M;D) = \mathcal{X}_\mathcal{F}(M;P) \cap \mathcal{X}(M;D)$$

and

$$\mathcal{X}_\mathcal{F}(M_i;D_i) = \mathcal{X}_\mathcal{F}(M_i;P_i) \cap \mathcal{X}(M_i;D_i)$$

(i=1,2), respectively. It is then straight-forward to show that

$$\mathcal{X}_\mathcal{F}(M;D_1,D_2) := \mathcal{X}_\mathcal{F}(M_1;D_1) \times \mathcal{X}_\mathcal{F}(M_2;D_2)$$

is a Lie subalgebra of the Lie algebra $\mathcal{X}_\mathcal{F}(M;D)$ and

$$\mathcal{X}_\mathcal{F}(M;D_1,D_2) = \mathcal{X}_\mathcal{F}(M;D_1) \oplus \mathcal{X}_\mathcal{F}(M;D_2),$$

where $\mathcal{X}_\mathcal{F}(M;D_1)$ is the set of all vector fields on M of the form $(X_1,0)$ with $X_1 \in \mathcal{X}_\mathcal{F}(M_1;D_1)$ and $\mathcal{X}_\mathcal{F}(M;D_2)$ the set of all vector fields on M of the form $(0, X_2)$ with $X_2 \in \mathcal{X}_\mathcal{F}(M_2;D_2)$.

If $\mathcal{F}(M,R;D_1,D_2)$ is the set of all functions $f \in \mathcal{F}(M,R;D)$ so that $X_f \in \mathcal{X}_\mathcal{F}(M;D_1,D_2)$ then we have

Proposition 4 : $\mathsf{F}(M,R;D_1,D_2)$ is contained in $\mathsf{F}(M,R;D)$.

Let $p^{\times}_{D*}$ be the restriction of p^{F}_{D*} to the set $\mathsf{X_F}(M;D_1,D_2)$. Then, $p^{\times}_{D*}$ is equal to the product maping, $p^{\times}_{D*} = p^{F}_{D_1*}\times p^{F}_{D_2*}$ and $\mathsf{K}(p^{\times}_{D*}) = \mathsf{K}(p^{F}_{D_1*})\times\mathsf{K}(p^{F}_{D_2*})$. Since $\mathsf{K}(p^{\times}_{D*}) = \mathsf{K}(p^{F}_{D*})\cap\mathsf{X_F}(M;D_1,D_2)$, then $\mathsf{K}(p^{\times}_{D*})\subset\mathsf{K}(p^{F}_{D*})$.

Let $\mathsf{X_F}^{\sim\times}(M;D_1,D_2)$ be the quotient algebra of $\mathsf{X_F}(M;D_1,D_2)$ relative to the kernel $\mathsf{K}(p^{\times}_{D*})$. Then $\mathsf{X_F}^{\sim\times}(M;D_1,D_2)$ is a Lie subalgebra of $\mathsf{X_F}^{\sim}(M;D)$.

Consider now the cartesian product $\mathsf{X_F}^{\sim}(M_1;D_1)\times\mathsf{X_F}^{\sim}(M_2;D_2)$ and define the bracket $[.,.]^{\times}$ in $\mathsf{X_F}^{\sim}(M_1;D_1)\times\mathsf{X_F}^{\sim}(M_2;D_2)$ according to

$$[([X_1],[X_2]),([Y_1],[Y_2])]^{\times} = ([[X_1],[Y_1]]^{\sim_1},[[X_2],[Y_2]]^{\sim_2})$$

for all pair $([X_1],[X_2]),([Y_1],[Y_2])$ in the set $\mathsf{X_F}^{\sim}(M_1;D_1)\times\mathsf{X_F}^{\sim}(M_2;D)$, where $[.,.]^{\sim_1}$ and $[.,.]^{\sim_2}$ is accordingly the quotient bracket in $\mathsf{X_F}^{\sim}(M_1;D_1)$ and $\mathsf{X_F}^{\sim}(M_2;D_2)$, respectively.

Proposition 5 : *There exists an injective Lie homomorphism from the Lie algebra* $(\mathsf{X_F}^{\sim}(M_1;D_1)\times\mathsf{X_F}^{\sim}(M_2;D_2),[.,.]^{\times})$ *into the Lie algebra* $(\mathsf{X_F}^{\sim}(M;D),[.,.]^{\sim})$ *whose image is the subalgebra* $\mathsf{X_F}^{\sim\times}(M;D_1,D_2)$ *of* $(\mathsf{X_F}^{\sim}(M;D),[.,.]^{\sim})$.

Consequently, $\mathsf{X_F}^{\sim}{}_c(M_1;D_1)\times\mathsf{X_F}^{\sim}{}_c(M_2;D_2)$ and $\mathsf{X_F}^{\sim}{}_c(M;D)$ is partially homomorphic and

$$\mathsf{X_F}^{\sim}{}_c(M_1;D_1)\oplus\mathsf{X_F}^{\sim}{}_c(M_2;D_2)\cong \mathsf{X_F}^{\sim\times}{}_c(M;D_1,D_2)\subset\mathsf{X_F}^{\sim}{}_c(M;D).$$

Let $\mathsf{F_K}(M,R;D_1,D_2)$ the set of all functions in $\mathsf{F}(M,R;D_1,D_2)$ which are contained in $\mathsf{K}(p^{\times}_{D*})$.

Proposition 6 : *Every function* $f\in\mathsf{F_K}(M,R;D_1,D_2)$ *can be written as the sum* $f = f_1?\pi_1 + f_2?\pi_2$, *where* $f_1\in\mathsf{F_K}(M_1,R;D_1)$ *and* $f_1\in\mathsf{F_K}(M_2,R;D_2)$.

Main Result : If $\mathsf{S}^{\times}{}_{GQ}(M;D)$ is defined as the semidirect sum

$$\mathsf{F_K}(M,R;D_1,D_2)\oplus_s\mathsf{X_F}^{\sim\times}{}_c(M;D_1,D_2),$$

then $\mathsf{S}^{\times}{}_{GQ}(M;D)\subset\mathsf{S}_{GQ}(M;D)$. The completeness of $\mathsf{S}^{\times}{}_{GQ}(M;D)$ is therefore necessary (but not sufficient) for the completeness of $\mathsf{S}_{GQ}(M;D)$. In turn, it means also that the completeness of $\mathsf{S}_{GQ}(M_1;D_1)$ and of $\mathsf{S}_{GQ}(M_1;D_1)$ is necessary (but not sufficient) for the completeness of $\mathsf{S}_{GQ}(M;D)$.

4. OUTLOOK

Implementing the above scheme to a physical system consisting of a massive particle with spin for instance leads to a kind of superselection rule, where the spin variables are eliminated automatically from the ideal $\mathsf{F_K}(R^6\times S^2,R;D_R,D_S)$ of $\mathsf{S}_{GQ}(R^6\times S^2;D)$, where

$$\mathsf{F_K}(R^6\times S^2,R;D_R,D_S) = \mathsf{F_K}(R^6\times S^2,R;D_R) \cong C^{\infty}(R^3,R).$$

The ideal is, as already stated above, associated with the localization of the particle in its configuration space. However, it would be the matter of investigation in [10]. Such result, of course, is in concomitant with the wide accepted understanding that spin is an internal degree of freedom.

Acknowledgment

The author would like to thank his colleges in the Laboratory of Atomic and Nuclear Physics, Department of Physics, Gadjah Mada University, for interesting discussion and wonderful atmosphere.

REFERENCES

Emch, G.G., 1984. *Mathematical and Conceptual Foundations of 20th-Century Physics (Notas de Mathemática 100)*, Elsevier Science Publishers B.V. Amsterdam.

Landsman, N. P., 1998. *Mathematical Topics Between Classical and Quantum Mechanics*. Springer-Verlag, Heidelberg.

Libermann, P., and C.-M. Marle, 1987. Symplectic *Geometry and Analytical Mechanics*, D. Reidel Publishing Company, Dordrecht.

Rosyid, M. F., 2003. On the Relation between Configuration and Phase Space Quantization I, *Journal of the Indonesian Mathematical Society* **9** (1).

Rosyid, M.F., 2005a. *On the Geometric Quantization of Poisson Manifolds,* In *Proceeding of National Seminar on Research, Eduaction, and Application of Natural Sciences,* Fakultas MIPA UNY.

Rosyid, M.F., 2005b. *The Classical and Quantum Dynamics on Cotangent Bundle Like Symplectic Manifolds*, Presented in *HFIY Meeting* at Universitas Sarjana Wiyata 9 April 2005, Yogyakarta.

Rosyid, M.F., 2005c. GQ-consistent Quantum Borel Kinematics, In the *Proceeding of International Conference on Applied Mathematics 2005 (ICAM05) Bandung, Indonesia*.

Rosyid, M.F., 2007a. On the Relation between Configuration and Phase Space Quantization II, Presented in the *International Conference on Mathematics and Its Applications, SEAMS –GMU.*

Rosyid, M.F., 2007b. GQ-consistent Quantum Borel Kinemtics : Unitary Representation of the Group of Canonical Transformations, Presented in *the International Conference on Mathematics and Its Applications, SEAMS–GMU*.

Simms, D.J., and N.M.J., Woodhouse, 1976. *Lectures on Geometric Quantization* (*Lecture Notes in Physics 53*), Springer-Verlag, New York.