

Simulation modeling of the sensitivity analysis of differential effects of the intrinsic growth rate of a fish population: its implication for the selection of a local minimum

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Abstract— *The vulnerability of the differential effects of the intrinsic growth rates of the fish population on the uncertainty analysis can only be controlled by using the mathematical technique of a sensitivity analysis that is called a local minimum selection method based on a Matlab numerical scheme of ordinary differential equations of order 45 (ODE 45). The quantification of the p-norms sensitivity analysis depends on the application of the 1-norm, 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity-norm. In the context of this study, the best-fit intrinsic growth rate of fish population with a small error has occurred when its value is 0.303 which minimizes the bigger sensitivity values previously obtained irrespective of the p-norm sensitivity values. The novel results which we have obtained have not been seen elsewhere. These results are fully presented and discussed in this study.*

Keywords— *Uncertainty analysis, differential effects, p-norms sensitivity analysis, intrinsic growth rate, local minimum, ODE45.*

I. INTRODUCTION

Following Ekaka-a et al (2012), sensitivity analysis is a measure of the resultant effect due to a variation of a model parameter value on the model solution trajectories. It is a mathematical tool to enhance model validation and prediction. On the other hand, on-going debate among many researchers including Palumbi (1999), Sumali (2002), Pitchford (2007) and Bohnsack (1993) has shown that marine reserve is not only economically important but can also serve as a tool for equitable management of biodiversity especially in the context of fisheries. Kar and Charkraborty (2009), in their work titled marine reserves and its consequences as a fisheries management tool

described a prey-predator type fishing model with prey dispersed in a 2-patch environment, one of which is called a free fishing zone and another, a protected zone. Their main method of investigation uses the simulation process. One key contribution from their work states that prey-predator dichotomy do not matter when implementation of a reserve is considered. Their second result shows that reserves will be most effective when coupled with fishing effort controls in adjacent fisheries. Despite the fact that marine reserves and its consequences can be effectively utilized as a fisheries management tool, it is still an open research problem that these authors did not consider the technique of sensitivity analysis which is vital numerical incentive in a decision process that can lead to an effective fisheries management.

It remains an open problem to study the differential effects of varying the intrinsic growth rate of the fish population on the uncertainty analysis using a one-at-a-time sensitivity analysis Hamby 1995. It is against this background that we propose to use ODE45 RungeKutta numerical scheme with initial condition $2 \ 4 \ 2$ over a period of fifty (50) weeks to study the differential effects of the intrinsic growth rate of fish population on the sensitivity analysis that is indexed by seven classifications of the sensitivity analysis, namely: 1-norm error analysis, 2-norm error analysis, 3-norm error analysis, 4-norm error analysis, 5-norm error analysis, 6-norm error analysis and infinity-norm error analysis.

II. MATHEMATICAL FORMULATIONS

The mathematical model for this research work is based on the published article by Kar and Charkraborty (2009) given by the first order non-linear ordinary differential equations

which describes the prey-predator fisheries management model. Hence, we modify the model as follows:

$$\frac{d}{dt}x(t) = r_1x(t) - \beta_1x(t)^2 - \mu_1x(t)y(t) - \frac{\sigma}{\alpha}x(t) + \frac{\sigma}{\alpha-1}y(t) - mx(t)z(t) \quad (3.1)$$

$$\frac{d}{dt}y(t) = r_2y(t) - \beta_2x(t)y(t) - \mu_2y(t)^2 + \frac{\sigma}{\alpha}x(t) - \frac{\sigma}{\alpha-1}y(t) - ny(t)z(t) - qEy(t) \quad (3.2)$$

$$\frac{d}{dt}z(t) = r_3z(t) - \frac{r_3\gamma z(t)^2}{x(t)+y(t)} \quad (3.3)$$

$$x(0) = 2, y(0) = 4, z(0) = 2 \quad (3.4)$$

In this model, $x(t)$ represents the fish stock in the reserved area at time t , $y(t)$ is the fish stock in the unreserved area at time t , $z(t)$ is the biomass density of the predator species at time t . r_1 is the intrinsic growth rate of fish stock x , r_2 is the intrinsic growth rate of fish stock y , r_3 is the intrinsic growth rate of the predator species, γ is the equilibrium ratio between prey biomass and predator biomass, m is maximum relative increase in predation to the reserved area or simply put, the contribution of z to inhibit the growth of fish stock (x) in the reserved area, n is maximum relative increase in predation to the unreserved area or simply put, the contribution of z to inhibit the growth of fish stock (y) in the unreserved area. The predation terms are therefore defined as mxz and nyz with respect to fish stocks in the reserved and unreserved area respectively. In addition, σ is the mobility coefficient, α is the size of the reserved area, $1 - \alpha$ is the size of the unreserved area, q is catch ability coefficient, E is effort applied for harvesting fish population in the unreserved area.

Further mathematical interpretation can be invoked to describe the interaction of the three equations. In equation (3.1), $\frac{d}{dt}x(t) = r_1x(t)$ shows that without interaction with other y and z species, the x population grows unboundedly as time increases. The same observation is made for the y and z populations. This is mathematically correct but practically unrealistic, hence the need for the interaction between the species. We can deduce the following interpretations term by term:

$\beta_1x(t)^2$ is the contribution of fish stock in the reserved area to inhibits its original exponential growth.

$\mu_1x(t)y(t)$ is the contribution of fish stock in the unreserved area to inhibit the growth of fish population in the reserved area.

$\frac{\sigma}{\alpha}x(t)$ is the effect of the ratio of the migration rate/mobility coefficient to the size of the reserve area to inhibit the growth of fish stock in the reserved area

$\frac{\sigma}{\alpha-1}y(t)$ is the effect of the ratio of the migration rate/mobility coefficient to the size of the unreserved area to enhance the growth of fish stock in the reserved area

$mx(t)z(t)$ is the contribution of z to inhibit the growth of fish stock on the reserve area

$\beta_2x(t)y(t)$ is the contribution of fish stock in the reserved area to inhibit the growth of fish population in the unreserved area.

$\mu_2y(t)^2$ is the contribution of fish stock in the unreserved area to inhibits its original exponential growth.

$\frac{\sigma}{\alpha}x(t)$ is the effect of the ratio of the migration rate/mobility coefficient to the size of the reserve area to inhibit the growth of fish stock in the unreserved area

$\frac{\sigma}{\alpha-1}y(t)$ is the effect of the ratio of the migration rate/mobility coefficient to the size of the unreserved area to inhibit the growth of fish stock in the unreserved area

$ny(t)z(t)$ is the contribution of the predator biomass to inhibit the growth of fish stock on the unreserved area

$qEy(t)$ is the contributed effect of the fishing effort and Catchability coefficient to inhibit the growth of the fish stock in the unreserved area.

$\frac{r_3\gamma z(t)^2}{x(t)+y(t)}$ is equilibrium ratio on the fish stocks in the reserved and unreserved area due to activities of the predator biomass.

For the purpose of our mathematical analysis, the value for model parameter r is the same value for: $r_1, r_2, \beta_1, \beta_2, \mu_1, \mu_2$ and $s = r_3$. In addition, we shall adopt the model parameter values as proposed by Kar and Charkraborty (2009): $r = 0.3, \sigma = 0.2, m = 1, n = 1, q = 0.0025, s = 0.1, \gamma = 0.95, E = 95$.

III. METHOD OF ANALYSIS

The core part of the algorithm which we have utilized to calculate the sensitivity of a model parameter is hereby described by the following steps which has also been implemented in the work of Ekaka-a et al (2013):

Step I: identify and code the control system of given model equations of continuous non-linear first order ordinary differential equation in which the model parameter is not varied. For the purpose of this analysis, the three solution trajectories are denoted by $x, y,$ and z .

Step II: identify and code a sub-model of the control system of given model equations of continuous non-linear

first order ordinary differential equation in which the model parameter is varied one-at-a-time. In this case, the three solution trajectories are denoted by x_m , y_m , and z_m .

Step III : code an appropriate Matlab program using ODE Runge-Kutta scheme to execute the program in Step I and Step II. With the initial conditions and a time range, the execution program will produce the solution trajectories for the programs in step I and step II.

On the execution program, specify the difference between the solution trajectories of the codes in step I and step II as $F_1 = x - x_m$, $F_2 = y - y_m$ and $F_3 = z - z_m$

Step IV: Use the execution program to calculate the 1-norm, 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity-norm for the three solution trajectories of the control model equations and similarly for the solution trajectories for the difference between the solution trajectories. For example, for the x and x_m solution trajectories which assume precise data points such as x_j and x_{mj} , where the subscript j takes on the values of 1, 2, 3, 4, ..., n, the 1-norm for the x solution trajectory is defined as the sum of the absolute values of x_1, x_2, x_3 , up to the n th point x_n . In the same manner, the 2-norm of the x_j solution trajectory is defined by the positive square root of the sum of the squares of absolute values of x_1, x_2, x_3 , up to the n th point x_n . Similarly, the 3-norm for the x solution trajectory is defined as the sum of the absolute values (cubed) of x_1, x_2, x_3 , up to the n th point x_n . The 4-norm for the x solution trajectory is defined as the sum of the absolute values (to fourth power) of x_1, x_2, x_3 , up to the n th point x_n . The 5-norm for the x solution trajectory is defined as the sum of the absolute values (to fifth power) of x_1, x_2, x_3 , up to the n th point x_n . The 6-norm for the x solution trajectory is defined as the sum of the absolute values (to sixth power) of x_1, x_2, x_3 , up to the n th point x_n . The infinity norm for x_m is defined by the maximum value of the set of the absolute values of x_1, x_2, x_3 , up to the n th point x_n . The same procedure holds for the calculation of 1-norm, 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity norm for y and z solution trajectories.

Step V: In the execution program, calculate the difference between the solution trajectories of the control model equation and the varied model equation by $F_1 = x - x_m$, $F_2 = y - y_m$ and $F_3 = z - z_m$ for the given range of data points when $j = 1, 2, 3, 4, \dots, n$ such that the difference between the solution trajectories of the control model equations and the varied model equations would be $F_1 = x_j - x_{mj}$, $F_2 = y_j - y_{mj}$ and $F_3 = z_j - z_{mj}$.

For the purpose of this analysis, we will also calculate the 1-norm, 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and

infinity norm of F_1, F_2 and F_3 . For example, 1-norm of F_1 will be the sum of the absolute values of the data points $(x_1 - x_{m1}), (x_2 - x_{m2}), (x_3 - x_{m3}), (x_4 - x_{m4}), \dots, (x_n - x_{mn})$ where x_1 and x_{m1} stand for the first data point of x solution trajectory and the first data point of the modified x_m solution trajectory respectively. x_2 and x_{m2} stand for the second data point of x solution trajectory and the second data point of the modified x_m solution trajectory respectively and so forth. The 1-norm of F_2 will be the sum of the absolute values of the data points $(y_1 - y_m), (y_2 - y_{m2}), (y_3 - y_{m3}), (y_4 - y_{m4}), \dots, (y_n - y_{mn})$ where y_1 and y_{m1} stand for the first data point of y solution trajectory and the first data point of the modified y_m solution trajectory respectively. y_2 and y_{m2} stand for the second data point of y solution trajectory and the second data point of the modified y_m solution trajectory respectively and so forth. Similarly, the 1-norm of F_3 will be the sum of the absolute values of the data points $(z_1 - z_{m1}), (z_2 - z_{m2}), (z_3 - z_m), (z_4 - z_{m4}), \dots, (z_n - z_{mn})$ where z_1 and z_{m1} stand for the first data point of z solution trajectory and the first data point of the modified z_m solution trajectory respectively. z_2 and z_{m2} stand for the second data point of z solution trajectory and the second data point of the modified z_m solution trajectory respectively and so forth. The 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity norm can similarly be calculated for the differences of three solution trajectories F_1, F_2 and F_3 .

Step VI: To calculate the cumulative percentage effect of variation of a chosen model parameter one-at-a-time when other parameters are fixed on each solution trajectory. For x solution trajectory, we will calculate the following values: (1-norm of F_1 divided by the 1-norm of x) multiplied by 100; (2-norm of F_1 divided by the 2-norm of x) multiplied by 100; (3-norm of F_1 divided by the 3-norm of x) multiplied by 100; (4-norm of F_1 divided by the 4-norm of x) multiplied by 100; (5-norm of F_1 divided by the 5-norm of x) multiplied by 100; (6-norm of F_1 divided by the 6-norm of x) multiplied by 100 and (infinity-norm of F_1 divided by the infinity-norm of x) multiplied by 100.

To calculate the percentage cumulative effect of variation of a model parameter one-at-a-time when other parameters are fixed on y solution trajectory, we will calculate the following values: (1-norm of F_2 divided by the 1-norm of y) multiplied by 100; (2-norm of F_2 divided by the 2-norm of y) multiplied by 100; (3-norm of F_2 divided by the 3-norm of y) multiplied by 100; (4-norm of F_2 divided by the 4-norm of y) multiplied by 100; (5-norm of F_2 divided by the 5-norm of y) multiplied by 100; (6-norm of F_2 divided by

the 6-norm of y) multiplied by 100 and (infinity-norm of F_2 divided by the infinity-norm of y) multiplied by 100.

To calculate the percentage cumulative effect of variation of a model parameter one-at-a-time when other parameters are fixed on z solution trajectory, we will calculate the following values: (1-norm of F_3 divided by the 1-norm of z) multiplied by 100; (2-norm of F_3 divided by the 2-norm of z) multiplied by 100; (3-norm of F_3 divided by the 3-norm of z) multiplied by 100; (4-norm of F_3 divided by the 4-norm of z) multiplied by 100; (5-norm of F_3 divided by the 5-norm of z) multiplied by 100; (6-norm of F_3 divided by the 6-norm of z) multiplied by 100 and (infinity-norm of F_3 divided by the infinity-norm of z) multiplied by 100.

Due to the unstable values of the 1-norm, 2-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity-norm specifications, we adopt to use a compact value for related percentage values of these norms. For example, the cumulative percentage value of 1-norm sensitivity in terms of the difference in solution trajectories involves the sum of the 1-norm values due to F_1 solution trajectory, 1-norm values due to F_2 solution trajectory and 1-norm values due to F_3 solution trajectory. The same procedure can be followed to calculate the cumulative percentage values for 3-norm, 3-norm, 4-norm, 5-norm, 6-norm and infinity-norm

sensitivities in terms of F_1 solution trajectory, F_2 solution trajectory and F_3 solution trajectory.

The cardinal points of sensitivity analysis results interpretation are:

- ✓ the parameter which when varied a little and produces the biggest cumulative effect on the solution trajectories is called a most sensitive parameter. And it is highly uncertain. This by implication will attract error in prediction and thus require further validation.
- ✓ the parameter which when varied a little and produces lower or least sensitivity values is called a less or least sensitive parameter and requires lesser validation.

Method for Best-fit parameter value selection

In order to select the best-fit parameter value for each model parameter, we recommend that

- ✓ At 100% variation, the coordinates of the solution trajectories have same values, sums, squares and square roots = 0. Therefore, the norm values at 100% are zero.

The Local minimum value is selected at a point where the smallest norm value occurs before or after the 100% variation.

IV. RESULTS AND DISCUSSION

On the application of the define method of analysis, we hereby present and discuss the following results:

Table.1: sensitivity analysis results for model parameter r

r	1-norm	2-norm	3-norm	4-norm	5-norm	6-norm	∞ -norm
0.0030	143.9520	78.8284	82.0093	66.2747	56.624	50.6594	29.8099
0.0060	143.1933	78.4409	81.6042	65.9501	56.348	50.4130	29.6754
0.0090	142.3997	78.0321	81.1766	65.6070	56.056	50.1521	29.5334
0.0120	141.5698	77.6010	80.7256	65.2447	55.747	49.8763	29.3837
0.0150	140.7023	77.1468	80.2503	64.8624	55.421	49.5850	29.2259
0.0180	139.7958	76.6685	79.7499	64.4593	55.077	49.2776	29.0605
0.0210	138.8491	76.1654	79.2236	64.0350	54.715	48.9538	28.8879
0.0240	137.8610	75.6368	78.6706	63.5887	54.334	48.6131	28.7078
0.0270	136.8304	75.0818	78.0903	63.1201	53.934	48.2553	28.5200
0.0300	135.7565	74.4999	77.4821	62.6287	53.514	47.8800	28.3243

What do we learn from Table 1?

We have observed from table 1 that as the value of the model parameter r increases monotonically from 0.003 (approx) to 0.03 (approx), the 1-norm error data decreases monotonically from the value of 143.95 (approx.) to 135.76 (approx.), the 2-norm error data decreases monotonically from the value of 78.83 (approx.) to 75.50 (approx.), the 3-norm error data decreases monotonically from the value of 82.01 (approx.) to 77.48 (approx.), the 4-norm error data

decreases monotonically from the value of 66.27 (approx.) to 62.63 (approx.), the 5-norm error data decreases monotonically from the value of 56.62 (approx.) to 53.51 (approx.), the 6-norm error data decreases monotonically from the value of 50.66 (approx.) to 47.88 (approx.) and the infinity-norm error data decreases monotonically from the value of 29.81 (approx.) to 28.32 (approx.).

On the basis of this present analysis we have observed that $r = 0.003$ is associated with relatively higher uncertainty when compared to the value of $r = 0.03$ irrespective of the type of p -norm we have used to calculate the sensitivity analysis. Despite the observed uncertainty analysis results due to a one percent to ten percent variation of the intrinsic growth rate of the fish population, it is clear that the statistical range of the p -norm sensitivity values are listed as follows: the range of 1-norm statistical range is 8.19, the

range of 2-norm statistical range is 3.33, the range of 3-norm statistical range is 4.53, the range of 4-norm statistical range is 3.64, the range of 5-norm statistical range is 3.11, the range of 6-norm statistical range is 2.78, the range of infinity-norm statistical range is 1.49.

These bigger sensitivity values which indicate high uncertainty of the intrinsic growth rate of the fish population can be further minimized.

Table.2: local minimum results for parameter r

r	1-norm	2-norm	3-norm	4-norm	5-norm	6-norm	∞ -norm
0.2880	4.3024	2.2917	2.4375	2.0011	1.739	1.5812	1.0897
0.2910	3.2128	1.7106	1.8194	1.4936	1.298	1.1803	0.8137
0.2940	2.1326	1.1350	1.2072	0.9910	0.861	0.7831	0.5401
0.2970	1.0617	0.5648	0.6007	0.4931	0.429	0.3897	0.2688
0.3000	0.0000	0.0000	0.0000	0.0000	0.000	0.0000	0.0000
0.3030	1.0527	0.5595	0.5951	0.4885	0.425	0.3860	0.2663
0.3060	2.0964	1.1138	1.1847	0.9724	0.845	0.7685	0.5304
0.3090	3.1314	1.6630	1.7688	1.4517	1.262	1.1473	0.7926
0.3120	4.1577	2.2071	2.3475	1.9265	1.674	1.5225	1.0527
0.3150	5.1755	2.7463	2.9209	2.3969	2.083	1.8943	1.3106

What do we learn from Table 2?

Table 2 shows a result of the cumulative effect of 96 to 105 percent variation of model parameter r . At 100percent variation, there are no changes in the original and the varied solution trajectories, hence the zero values.

Observe that as the model parameter value increase monotonically from a low value of 0.2880 to 0.0000 corresponding to 100 percent variation and then increases monotonically to 0.3150 approximately in column I, the 1-norm sensitivity value decreased monotonically from a value of 4.30 to 0.0000 corresponding 100 percent variation and then increases monotonically to 5.18 approximately in column II, the 2-norm sensitivity value decreased monotonically from a value of 2.29 to 0.0000 corresponding 100percent variation and then increases monotonically to 2.75 approximately in column III, the 3-norm sensitivity value decreased monotonically from a value of 2.44 to 0.0000 corresponding 100percent variation and then increases monotonically to 2.92 approximately in column IV, the 4-norm sensitivity value decreased monotonically from a value of 2.00 to 0.0000 corresponding 100percent variation and then increases monotonically to 2.40 approximately in column V, the 5-norm sensitivity value decreased monotonically from a value of 1.74 to 0.0000 corresponding 100percent variation and then increases monotonically to 2.08 approximately in column VI, the 6-

norm sensitivity value decreased monotonically from a value of 1.58 to 0.0000 corresponding 100percent variation and then increases monotonically to 1.90 approximately in column VII, the infinity-norm sensitivity value decreased monotonically from a value of 1.09 to 0.0000 corresponding 100percent variation and then increases monotonically to 1.31 approximately in column VIII.

The local minimum is selected at the greatest lower bound of the data points for each p -norm results. For instance, the local minimum value for model parameter r is 0.303 where 1-norm local minimum value = 1.0527, 2-norm local minimum value = 0.5595, 3-norm local minimum value = 0.5951, 4-norm local minimum value = 0.4885, 5-norm local minimum value = 0.4250, 6-norm local minimum value = 0.3860 and infinity-norm local minimum value = 0.2663.

V. CONCLUSION

This present study has been able to reduce the uncertainty in the intrinsic growth rate of a fish population due to its variations using a combination of a numerical scheme and the mathematical p -norms. The present results compliment the earlier contribution of Ekaka-a et al (2012) that only considered the sensitivity analysis in the context of a shorter experimental time. This present proposed numerical scheme can be extended to study the sensitivity analysis of other

model parameter values which we did not consider in this study in our future investigation.

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