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Deterministic Stabilization of a Dynamical System using a Computational Approach

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Abstract— *The qualitative behavior of a multi-parameter dynamical system has been investigated. It is shown that changes in the initial data of a dynamical system will affect the stabilization of the steady-state solution which is originally unstable. It is further shown that the stabilization of a five-dimensional dynamical system can be used as an alternative method of verifying qualitatively the concept of the stability of a unique positive steady-state solution. These novel contributions have not been seen elsewhere; these are presented and discussed in this paper.*
Keywords— Deterministic, stabilization, dynamical system, steady-state solution, changing initial data.

I. INTRODUCTION

Agarwal and Devi (2011) studied in detail the mathematical analysis of a resource-dependent competition model using the method of local stability analysis. Other relevant mathematical approaches to the concept of stability analysis have been done [Rescigno (1977);Hallam, Clark and Jordan (1983);Hallam, Clark and Lassiter (1983);Hallam and Luna (1984);Freedman and So (1985); Lancaster and Tismenetssky (1985);De Luna and Hallam (1987); Zhien and Hallam (1987);Freeman and Shukla (1991);Huaping and Zhien (1991);Garcia-Montiel and Scatena (1994); Chattopadhyay (1996);Hsu and Waltman (1998); Dubey and Hussain (2000);Hsu, Li and Waltman (2000);Thieme (2000); Shukla, Agarwal, Dubey and Sinha (2001); Ekaka-a (2009); Shukla, Sharma, Dubey and Sinha (2009); Yan and Ekaka-a (2011);Dhar, Chaudhary and Sahu (2013); Akpodee and Ekaka-a (2015)]. The method of thispresent study uses the technique of a numerical simulation to quantify the qualitative characteristics of a complex dynamical system with changing initial data.

II. MATHEMATICAL FORMULATION

We have considered the following continuous multi-parameter system of nonlinear first order ordinary differential equations indexed by the appropriate initial conditions according to Agarwal and Devi (2011):

$$\frac{dx_1}{dt} = a_1x_1 - a_2x_1^2 - \alpha x_1x_2 + \alpha_1x_1R - k_1\delta_1x_1T,\ x_1(0) = x_{10} \geq 0, \quad (1a)$$

$$\frac{dx_2}{dt} = b_1x_2 - b_2x_2^2 - \beta x_1x_2 + \beta_1x_2R - k_2\delta_2x_2T,\ x_2(0) = x_{20} \geq 0, \quad (1b)$$

$$\frac{dR}{dt} = c_1R - c_2R^2 - \alpha_1x_1R - \beta_1x_2R - k\gamma RT,\ R(0) = R_0 \geq 0, \quad (1c)$$

$$\frac{dP}{dt} = \eta x_1 + \eta x_2 - (\lambda_0 + \theta)P,\ P(0) = P_0 \geq 0, \quad (1d)$$

$$\frac{dT}{dt} = Q + \mu\theta P - \delta_0T - \delta_1x_1T - \delta_2x_2T - \gamma RT, \quad T(0) = T_0 \geq 0, \quad (1e)$$

where

x_1 and x_2 are the densities of the first and second competing species, respectively, R is the density of resource biomass, P is the cumulative concentration of precursors produced by species forming the toxicant, T is the concentration of the same toxicant in the environment under consideration, Q is the cumulative rate of emission of the same toxicant into the environment from various external sources, a_1 and b_1 are the intrinsic growth rates of the first and second species, respectively, a_2 and $b_2$are intraspecific interference coefficients,α, β are the interspecific interference coefficients of first and second species, respectively,α_1 and β_1 are the growth rate coefficients of first and second species, respectively due to resource biomass. k_1, k_2 and k are fractions of the assimilated amount directly affecting the growth rates of densities of competing species and resource biomass, η is the growth rate coefficient of the cumulative concentration of precursors. λ_0 is its depletion rate coefficient due to natural factors whereas θ is the depletion rate coefficient caused by its transformation into the same toxicant of concentration T. μ is the rate of the formation of the toxicant from precursors of competing species. δ_1, δ_2 and γ are the rates of depletion of toxicant in the environment due to uptake of toxicant by species and their resource biomass, respectively.

It is assumed that the resource biomass grows logistically with the supply rate of the external resource input to the system by constant c_1 and its density reduces due to certain

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degradation factors present in the environment at a rate c_2.It is further assumed that the toxicant in the environment is washed out or broken down with rate δ_0.

Research Question

For the purpose of this study, we have considered the following vital research question:How does a given dynamical multi-parameter system of continuous nonlinear first-order ordinary differential equations respond to a qualitative characteristic, that is, assuming a point $(x_{1e}, x_{2e}, R_e, P_e, T_e)$ is an arbitrary steady-state solution, as the independent variable t approaches infinity $(t \to \infty)$, do $x_1 \to x_{1e}$, $x_2 \to x_{2e}$, $R \to R_e$, $P \to P_e$, $T \to T_e$ under some simplifying initial conditions? This mathematical idea is a necessary and sufficient condition that quantifies the concept of the stabilization of the steady-state solution $(x_{1e}, x_{2e}, R_e, P_e, T_e)$ (Yan and Ekaka-a, 2011). In other words, for a complex system of nonlinear first-order ordinary differential equations whose interacting functions are continuous and partially differentiable, what is the likely qualitative characteristic of such a system? The focus of this chapter will tackle the following proposed problem that is clearly defined next.

Research Hypothesis

It is a well-established ecological fact that the initial ecological data, which mathematicians called initial conditions, are not static characteristic values of a dynamical system. The corresponding core research question is, when the initial data change, how does the dynamical system respond to this change over a longer period of time? This hypothesis if successfully tested and proved in this research, has the potential to provide an insight in the further study of ecosystem stability and ecosystem planning.

Method of Analysis

A well-defined MATLAB ODE45 function has been used to construct tables to determine the effect of changing values of initial data on the stability of the dynamical system for large values of the independent variable t. Following Agarwal and Devi (2011), the values of parameter values which are used in the numerical simulations for system (1) are:

$a_1 = 5$, $a_2 = 0.22$, $\alpha = 0.007$, $\alpha_1 = 0.2$, $k_1 = 0.1$, $\delta_1 = 0.05$, $b_1 = 3$, $b_2 = 0.26$, $\beta = 0.008$, $\beta_1 = 0.04$, $k_2 = 0.2$, $\delta_2 = 0.04$, $\eta = 0.5$, $\lambda_0 = 0.01$, $\theta = 3$, $\mu = 0.2$, $\delta_0 = 7$, $\gamma = 0.3$, $c_1 = 10$, $c_2 = 0.3$, $k = 0.1$, $Q = 30$.

III. RESULTS AND DISCUSSIONS

Some twenty (20) numerical simulations are observed to determine the effect of changing values of initial data on the stability of the dynamical system for $t = 3650$ days as shown in Table 1 below:

Table.1: Numerical simulation of a dynamical system for changing initial data at $t = 3650$ days, using a MATLAB ODE45 numerical scheme.

Example	Initial Data (ID)	Independent Variable t days	x_{1e}	x_{2e}	R_e	P_e	T_e
1	1	3650	25.4443	15.2308	30.7270	6.7195	2.1454
2	2	3650	25.3091	15.2308	30.6851	6.7195	2.1086
3	3	3650	25.3551	15.2308	30.6872	6.7195	2.1054
4	4	3650	25.3783	15.2308	30.6901	6.7195	2.1042
5	5	3650	25.3923	15.2308	30.6931	6.7195	2.1041
6	6	3650	25.4018	15.2308	30.6958	6.7195	2.1046
7	7	3650	25.4085	15.2308	30.6983	6.7195	2.1055
8	8	3650	25.4129	15.2308	30.6979	6.7195	2.1066
9	9	3650	25.4169	15.2308	30.6997	6.7195	2.1080
10	10	3650	25.4202	15.2308	30.7014	6.7195	2.1094
11	11	3650	25.4229	15.2308	30.7030	6.7195	2.1108
12	12	3650	25.4252	15.2308	30.7045	6.7195	2.1123
13	13	3650	25.4272	15.2308	30.7058	6.7195	2.1138
14	14	3650	25.4288	15.2308	30.7071	6.7195	2.1153
15	15	3650	25.4303	15.2308	30.7083	6.7195	2.1168
16	16	3650	25.4316	15.2308	30.7094	6.7195	2.1183

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Example	Initial Data (ID)	Independent Variable t days	x_{1e}	x_{2e}	R_e	P_e	T_e
17	17	3650	25.4328	15.2308	30.7105	6.7195	2.1197
18	18	3650	25.4338	15.2308	30.7115	6.7195	2.1211
19	19	3650	25.4347	15.2308	30.7125	6.7195	2.1225
20	20	3650	25.4356	15.2308	30.7134	6.7195	2.1239

where
ID 1 = (2, 0.01, 0.01, 0.1, 0.1), ID 2 = (0.10, 0.01, 0.01, 0.1, 0.1),
ID 3 = (0.15, 0.01, 0.01, 0.1, 0.1), ID 4 = (0.20, 0.01, 0.01, 0.1, 0.1),
ID 5 = (0.25, 0.01, 0.01, 0.1, 0.1), ID 6 = (0.30, 0.01, 0.01, 0.1, 0.1),
ID 7 = (0.35, 0.01, 0.01, 0.1, 0.1), ID 8 = (0.40, 0.01, 0.01, 0.1, 0.1),
ID 9 = (0.45, 0.01, 0.01, 0.1, 0.1), ID 10 = (0.50, 0.01, 0.01, 0.1, 0.1),
ID 11 = (0.55, 0.01, 0.01, 0.1, 0.1), ID 12 = (0.60, 0.01, 0.01, 0.1, 0.1),
ID 13 = (0.65, 0.01, 0.01, 0.1, 0.1), ID 14 = (0.70, 0.01, 0.01, 0.1, 0.1),
ID 15 = (0.75, 0.01, 0.01, 0.1, 0.1), ID 16 = (0.80, 0.01, 0.01, 0.1, 0.1),
ID 17 = (0.85, 0.01, 0.01, 0.1, 0.1), ID 18 = (0.90, 0.01, 0.01, 0.1, 0.1),
ID 19 = (0.95, 0.01, 0.01, 0.1, 0.1), ID 20 = (1, 0.01, 0.01, 0.1, 0.1).

Considering Table 1, we deduced mathematically that, as $t \to \infty$ for the given initial conditions, $x_1 \to x_{1e}$, $x_2 \to x_{2e}$, $R \to R_e$, $P \to P_e$, $T \to T_e$. We have shown that as the initial data are changing, the system is approaching its steady-state. This shows that changes in the initial data of a dynamical system will affect the stabilization of the steady-state solution which is originally unstable.

Table.2: Test of stability of steady-state solutions for changing values of initial data, using a MATLAB ODE45 numerical scheme.

Example	Initial Data (ID)	Steady-state solution (or point)	λ_1	λ_2	λ_3	λ_4	λ_5	TOS
1	1	1	-18.1705	-9.3804	-5.7543	-3.0150	-4.0180	Stable
2	2	2	-18.1498	-9.3527	-5.6958	-3.0150	-4.0180	Stable
3	3	3	-18.1527	-9.3547	-5.7159	-3.0150	-4.0185	Stable
4	4	4	-18.1547	-9.3568	-5.7260	-3.0150	-4.0186	Stable
5	5	5	-18.1563	-9.3589	-5.7321	-3.0150	-4.0187	Stable
6	6	6	-18.1576	-9.3607	-5.7362	-3.0150	-4.0187	Stable
7	7	7	-18.1587	-9.3623	-5.7391	-3.0150	-4.0186	Stable
8	8	8	-18.1588	-9.3621	-5.7411	-3.0150	-4.0187	Stable
9	9	9	-18.1596	-9.3633	-5.7428	-3.0150	-4.0187	Stable
10	10	10	-18.1604	-9.3644	-5.7442	-3.0150	-4.0187	Stable
11	11	11	-18.1610	-9.3654	-5.7453	-3.0150	-4.0186	Stable
12	12	12	-18.1616	-9.3664	-5.7463	-3.0150	-4.0186	Stable
13	13	13	-18.1622	-9.3672	-5.7472	-3.0150	-4.0186	Stable
14	14	14	-18.1627	-9.3680	-5.7478	-3.0150	-4.0185	Stable
15	15	15	-18.1632	-9.3688	-5.7485	-3.0150	-4.0185	Stable
16	16	16	-18.1637	-9.3695	-5.7490	-3.0150	-4.0185	Stable
17	17	17	-18.1641	-9.3701	-5.7495	-3.0150	-4.0185	Stable
18	18	18	-18.1645	-9.3708	-5.7500	-3.0150	-4.0184	Stable
19	19	19	-18.1649	-9.3714	-5.7505	-3.0150	-4.0184	Stable
20	20	20	-18.1653	-9.3720	-5.7507	-3.0150	-4.0184	Stable

where
Point 1 = (25.4443, 15.2308, 30.7270, 6.7195, 2.1454),
Point 2 = (25.3091, 15.2308, 30.6851, 6.7195, 2.1086),
Point 3 = (25.3551, 15.2308, 30.6872, 6.7195, 2.1054),
Point 4 = (25.3783, 15.2308, 30.6901, 6.7195, 2.1042),
Point 5 = (25.3923, 15.2308, 30.6931, 6.7195, 2.1041),
Point 6 = (25.4018, 15.2308, 30.6958, 6.7195, 2.1046),
Point 7 = (25.4085, 15.2308, 30.6983, 6.7195, 2.1055),
Point 8 = (25.4129, 15.2308, 30.6979, 6.7195, 2.1066),
Point 9 = (25.4169, 15.2308, 30.6997, 6.7195, 2.1080),
Point 10 = (25.4202, 15.2308, 30.7014, 6.7195, 2.1094),
Point 11 = (25.4229, 15.2308, 30.7030, 6.7195, 2.1101),
Point 12 = (25.4252, 15.2308, 30.7045, 6.7195, 2.1123),
Point 13 = (25.4272, 15.2308, 30.7058, 6.7195, 2.1138),
Point 14 = (25.4288, 15.2308, 30.7071, 6.7195, 2.1153),
Point 15 = (25.4303, 15.2308, 30.7083, 6.7195, 2.1168),
Point 16 = (25.4316, 15.2308, 30.7094, 6.7195, 2.1183),
Point 17 = (25.4328, 15.2308, 30.7105, 6.7195, 2.1197),
Point 18 = (25.4338, 15.2308, 30.7115, 6.7195, 2.1211),
Point 19 = (25.4347, 15.2308, 30.7125, 6.7195, 2.1225),
Point 20 = (25.4356, 15.2308, 30.7134, 6.7195, 2.1239).

What do we learn from Table 2? On the basis of this sophisticated computational approach which we have not seen elsewhere, we hereby infer that the stabilization of a five-dimensional dynamical system can be used as an alternative method of verifying qualitatively the concept of the stability of a unique positive steady-state solution which could have been a daunting task to resolve analytically.

However, this key contribution is only valid as long as the intrinsic growth rate a_1 is bigger than the intra-competition coefficient a_2 of the first competing species; the intrinsic growth rate b_1 is bigger than the intra-competition coefficient b_2 of the second competing species and the intrinsic growth rate of the resource biomass c_1 is bigger than the intra-competition coefficient c_2 of the resource biomass. In the event that the intra-competition coefficients of these three populations outweigh their corresponding intrinsic growth rates, will the specified steady-state solutions still be stable? Without loss of generality, it is interesting to observe that each of the twenty (20) stable steady-state solutions is also qualitatively well-defined within the choice of the model dynamics in the absence of proper model parameter estimation. The idea is consistent with the earlier discovery of Ekaka-a (2009).

IV. CONCLUSION AND RECOMMENDATION

We have shown in this research that stabilization is an alternative way of testing for stability. Therefore, the application of a computational approach in the determination of the stability characteristic using the concept of stabilization is one of the contributions of this work that can be used to move the frontier of knowledge in the field of numerical mathematics with respect to stability of a dynamical system.We recommend a further investigation of the effect of fixed initial data for changing values of the independent variable.

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