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Certain Generalized Prime elements

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Abstract— *In this paper we study different generalizations of prime elements and prove certain properties of these elements.*
Keywords— Prime, primary elements,weakly prime elements,weakly primary elements, 2-absorbing, 2-potent elements

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I. INTRODUCTION

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element a∈ L is called proper if a < 1. A proper element p of L is said to be prime if ab≤ p implies a ≤ p or b ≤ p. If a∈ L, b ∈ L,

(a : b) is the join of all elements c in L such that cb≤ a. A properelement p of L is said to be primary if ab≤ p implies a ≤ p orb^n≤ p for some positive integer n. If a∈ L, then $\sqrt{a}$ = ∨{x ∈ L_*/ x^n≤ a, n ∈ Z+}. An element a ∈ L is called a radical element ifa = $\sqrt{a}$. An element a∈ L is called compact if a $a \leq \vee_\alpha b_\alpha$ implies a ≤ $b_{\alpha_1} \vee b_{\alpha_2} \vee \ldots \vee b_{\alpha_n}$ for some finite subset $\{\alpha_{1,} \alpha_2 \ldots \alpha_n\}$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact element is compact. We shall denote by L_*, the set of compact elements of L.

An element i∈ L is called 2-absorbing element if abc≤i impliesab≤i or bc≤i or ca≤i. A proper element i∈ L is called 2-absorbing primary if for all a, b, c ∈ L, abc≤i implies either ab≤i or bc≤$\sqrt{i}$ or ca≤$\sqrt{i}$. This concept was defined by U.Tekir et.al. in [7]. It is observed that every prime element is 2-absorbing. An element i∈ L is called semi-prime if i =$\sqrt{i}$. An element is i called 2-potent prime if ab ≤i^2 implies a ≤i or b ≤i. (See [6]). Every 2-absorbing element of L is a 2-absorbing primary element of L. But the converse is not true. The element q = (12) is a 2-absorbing primary element of L but not 2-absorbing element of L. Also every primary element of L is a 2 absorbing primary element. But the converse is not true. The element q = (6) is a 2 absorbing primary element of L but not a primary element of L, since L is lattice of ideals of the ring. R =< Z,+, .>. For all these definition one can refer [1],[4],[5].

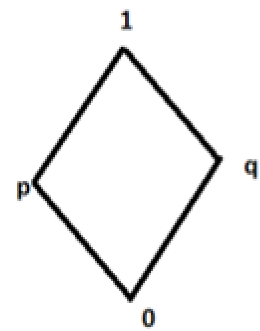
II. PRIME AND PRIMARY ABSORBING ELEMENTS

The concept of primary 2-absorbing ideals was introduced by Tessema et.al. [5]. We generalize this concept for multiplicative lattices.

An element i∈ L is said to be weakly prime if 0 ≠ ab≤i implies a ≤i or b ≤i.

It is easy to show that every prime element is 2- absorbing.

Ex. The following table shows multiplication of elements in themultiplicative lattice L = 0, p, q, 1.



.	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	p	0
q	0	q	0	q

In the above diagram 0, p, q are 2-absorbing.

The concept of 2-absorbing primary ideals is defined by A. Badawi,U. Tekir, E. Yetkin in [6]. The concept was generalized for multiplicative lattices by F. Calliap, E. Yetkin, and U. Tekir [8]. Weslightly modified this concept and defined primary 2-absorbing element.

Def.(2.1) An element i∈ L is said to be weakly 2-absorbing if0 ≠ abc≤i implies ab≤i or bc≤i or ca≤i. (See [7]).

Def.(2.2) An element i of L is called primary 2-absorbing ifabc≤i implies ab≤i or bc ≤$\sqrt{i}$ or ca≤$\sqrt{i}$, for all a, b, c ∈ L.

Ex. Every 2- absorbing element of L is primary 2-absorbing.

We obtain now the relation between primary 2-absorbing element and 2-absorbing element.

Theorem (2.3) Ifi is semi-prime and primary 2-absorbing element of a lattice L, then i is 2-absorbing.

Proof:- Suppose i is primary 2-absorbing. Let abc≤i. Then ab≤i or bc≤$\sqrt{i}$ or ca≤$\sqrt{i}$ where i= $\sqrt{i}$. Therefore i is 2-absorbing.

Theorem(2.4) If i is semi-prime and 2-potent prime element of Lthen i is prime.

Proof:- Let ab≤i. Then $(ab)^2$ = a^2b^2≤i^2. Then a^2≤i or b^2≤i,since i is 2-potent prime. This implies that a ≤$\sqrt{i}$ or b

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$\leq\sqrt{i}$. Buti being semi-prime, a ≤i or b ≤i. Hence iis a prime element.

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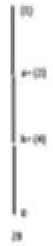
we note that :A prime element is 2-potent prime.



Consider the lattice L of ideals of ring R =<Z_8,+, . >. Then the only ideals of R are principal ideals (0), (2), (4), (1). Clearly, L = (0), (2), (4), (1) is compactly generated multiplicative lattice. The element (4) ∈ L is not prime but it is 2-potent prime.

Remark(2.5) If i is semi-prime and primary then i is prime.

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Now we establish the relation between 2-potent prime and primary element.

Theorem (2.6) Leti be a 2-potent prime. Then iis almostprimary if and only if i is primary.

Proof:- Let i be a primary element and ab≤i, ab$\nleq i^2$. Then a ≤I or b $\leq\sqrt{i}$. So i is almost primary. Conversely, let i be an almostprimary element. Assume that ab≤i. If ab$\nleq i^2$, then a ≤i orb $\leq\sqrt{i}$. Suppose ab≤ i^2. Then a ≤i or b ≤i≤ $\sqrt{i}$, since i is2-potent prime. Therefore iis primary.

We obtain the relation between semiprime and 2-absorbing element.

Theorem(2.7) If i is semi-prime 2-potent prime element of L,then i is 2-absorbing.

(Proof:-) Let abc≤i. Then $(abc)^2 = (ab)^2c^2 \leq i^2$. So $(ab)^2\leq$i or$c^2\leq$i, since i is 2-potent prime. As i is semi-prime, ab$\leq\sqrt{i}$ = i or c $\leq\sqrt{i}$ = i. Hence ab≤i or bc≤i, ac ≤i and i is 2-absorbing.

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Remark (2.8) A 2-absorbing primary element need not be 2-potent prime.

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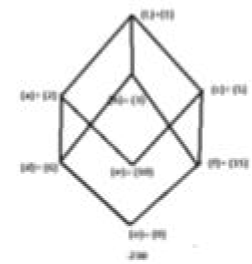
Ex.-Consider L as in exampleZ_{30}, the element (6) ∈ L is not2-potent prime.



The next result gives the condition for a semi-prime element tobe almost primary.

Theorem (2.9) Leti be a semi-prime element of L. Then iisalmost primary if i is weakly prime.

Proof:- Suppose i is weakly prime. Let ab≤i, ab$\nleq i^2$. Now ab$\nleq i^2$implies ab ≠ 0. Hence a ≤i or b ≤i= $\sqrt{i}$.

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Now we obtain a relation between primary element and weaklyprime element.

Theorem(2.10)Let i be a semi-prime element. Then iis primaryif and only if i is weakly prime.

Proof:- Assume that i is primary and 0 ≠ ab≤i. Then $a^n\leq$i forsome n ∈ Z_+ or b ≤i. If b ≤i, we are done.If an≤i, then a $\leq\sqrt{i}$ = i, since i is semi-prime. Hence iis weaklyprime. Suppose iis weakly prime. Let ab≤i. If 0 ≠ ab, then a ≤i or b ≤i$\leq\sqrt{i}$ and i is primary. If ab = 0 ≤i, a = 0 or b = 0 as L has no zero divisor. So a ≤i or b $\leq\sqrt{i}$ and i is primary.

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Theorem (2.11) If an element i is both weakly primary andsemi-prime, then i is weakly prime.

Proof:- Let 0 ≠ ab≤i. Then $a^n\leq$i or b ≤i, since i weaklyprimary. If b ≰i, then an ≤i implies a ≤i= $\sqrt{i}$. Hence i is weaklyprime.

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We now obtain a characterization of a weakly primary element.

Theorem (2.12) An element i is weakly primary if and only if(i : x) ≤ $\sqrt{i}$ or (i : x) = (i^2 : x) for all x ≰i.

Proof:- Let i be weakly primary and x ≰i. Since $i^2\leq$i, we have(i^2 : x) ≤ (i : x). Let y ≤ (i : x), then yx≤i. If yx = 0, thenyx≤ i^2 which implies y ≤ (i^2 : x). So in this case (i : x) = (i^2 : x).

Suppose yx≠ 0. Then yx≤i, x ≰i and i is weakly primary together imply $y^n\leq$i for some n ∈ Z+. Hence y $\leq\sqrt{i}$ and(i : x) $\leq\sqrt{i}$. Conversely suppose (i : x) $\leq\sqrt{i}$ or (i : x) = (i^2 : x)whenever x ≰i. Let 0 ≠ ab≤i. If b ≰i, we have nothing to prove. Otherwise b ≤i and obviously a ≤ (i : b).

Case 1 If a ≤ (i : b) ≤ $\sqrt{i}$, i is weakly primary.

Case 2) Suppose (i : b) = (i^2 : b) and (i : b) $\nleq\sqrt{i}$. In this case,there exists z ≤ (i : b) but z $\nleq\sqrt{i}$. Therefore $z^n\nleq$i for all n ∈ Z+.

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Now z ≤ (i : b) implies zb≤i for b ≰i. In particular, for b = z,z^2≤i, which is a contradiction. Hence the second case does notarises.

Theorem(2.13) Let i and j be distinct weakly prime elementsof L. Then (i∧ j) is weakly 2- absorbing.
Proof:- Let 0 ≠ abc≤ (i∧ j). Then abc≤i and abc≤ j. Since I and j are weakly prime elements, we have ab≤i or c ≤i and ab≤ j or c ≤ j.
Case 1) If ab≤i and ab≤ j we have ab≤ (i∧ j).
Case 2) If ab≤i and c ≤ j, then a ≤i or b ≤i and c ≤ j, since0 ≠ ab≤i and i is weakly prime. Thus ac ≤i, ac ≤ j or bc≤i andbc≤ j. This shows that ac ≤ (i∧ j) or bc≤ (i∧ j).
Case 3) Let c ≤i and ab≤ j. This case is similar to case (2).
Case 4) Suppose c≤i and c ≤ j. In this case ac ≤ j, ac ≤I and thus ac ≤ (i∧ j) together imply (i∧ j) is weakly 2-absorbing.

Next we have a property of a weakly prime element.
Theorem (2.14) Leti be a weakly prime element of L. Then I is weakly 2-absorbing.
Proof:- Let 0 ≠ abc≤i. Then a ≤i or bc≤i. This again impliesa ≤i or b ≤i or c ≤i. Hence ab≤i or bc ≤i or ac ≤i and i isweakly 2-absorbing.

III. TWIN ZERO AND WEAKLY PRIME ELEMENTS

The concept of a Twin zero of an ideal in a commutative rings withunity is introduced and studied in detail by A.Badawi et.al. [].
Wegeneralize this concept for multiplicative lattice and obtain some
results relating to this concept.
Definition 3.1) Let L be a multiplicative lattice and i∈ L,wesay that (a, b) is a twin zero of i if ab = 0, a ≰i, b ≰i.

Remark 3.2) If i is weakly prime element of L that is not a primeelement then i has twin zero (a,b) for some a, b ∈ L.

Theorem 3.3) Let i be a weakly prime element of L and supposethat (a, b) is a twin zero of i for some a, b ∈ L. Then ai = bi = 0.
Proof:- Suppose ai ≠ 0. Then there exists c ≤i such that ac ≠ 0. Hence a(b∨c) ≠ 0. Since (a, b) is a twin zero of i and ab = 0, we have a ≰i and b ≰i. As a ≰i, i is weakly prime and 0 ≠ a(b∨c) ≤ ac ≤i. We must have b ≤ c ≤i .Hence b ≤i, a contradiction. Hence ai =0 and similarly it can be shown that bi = 0.

Theorem 3.4) Leti be a weakly prime element of L. If i is notprime then i^2 = 0.
Proof:-Let (a, b) be twin zero of i. Hence ab =0, where a ≰i and b ≰i. Assume that i^2≠ 0. Suppose $i_1.i_2$≠ 0 for some i_1,i_2≤i. Then (a ∨i_1)(b ∨i_2) = $i_1.i_2$≠ 0 (by Theorem 3.3). Since 0 ≠(a ∨i_1)(b ∨i_2) ≤i and i is weakly prime, it follows that (a ∨i_1) ≤ior (b ∨i_2) ≤ i. Thus a ≤ i or b ≤i, which is a contradiction.Therefore i^2 = 0.

Theorem 3.5) Leti be a weakly prime element of L. If i is notprime then i≤$\sqrt{0}$ and i$\sqrt{0}$ = 0.
Proof:- Suppose i is not prime. Then by Theorem (3.4), i^2 = 0 and hence i≤$\sqrt{0}$. Let a = $\sqrt{0}$. If a ≤i, then ai = 0, by Theorem(3.4). Now assume that a ≰i and ai≠ 0. Hence ab≠ 0 for some b ≤i. Let m be the least positive integer such that a^m = 0. Since a(a^{m-1} ∨ b) = ab ≠ 0 and a ≰i, we have, (a^{m-1} ∨ b) ≤i. Since 0 ≠ a^{m-1} ≤ i and i is weakly prime, we conclude that a ≤i, a contradiction. Thus ai =0 for all a ≤$\sqrt{0}$. Therefore i$\sqrt{0}$ = 0.

Theorem 3.6) Let i be a weakly prime element of L and suppose(a, b) is twin zero of i. If ar ≤i for some r ∈ L, then ar = 0.
Proof:- Suppose 0 ≠ ar ≤i for some r ∈ L. Since (a, b) is twinzero of i, ab =0 where a ≰i and b ≰i. As i is weakly prime and0 ≠ ar ≤ i, it follows that r ≤i. By Theorem (3.3), ai = bi = 0.Hence r ≤i implies ar = 0, a contradiction. Therefore ar =0.

Theorem 3.7) Let i be a weakly prime element of L. Supposeab ≤i for some a, b ∈ L. If i has twin zero a_1,b_1 for some a_1≤ aand b_1≤ b then ab =0.
Proof:- Suppose (a_1,b_1) is a twin zero of i for some a_1 ≤ a and b_1≤ b and assume that ab ≠ 0. Hence cd ≠ 0 for some c ≤a andd ≤ b. Now 0 ≠ cd ≤ ab ≤i, where i is weakly prime. Hence c ≤ior d ≤i. Without loss of generality, we may assume that c ≤i. ByTheorem (3.4), i^2 = 0. If d ≤i then c ≤ i implies cd ≤i^2 = 0 andhence cd = 0, a contradiction. Therefore d ≰i. Next ab ≤ i, d ≤ bimplies ad ≤i. Also a_1≰i gives a ≰i. As i is weakly prime a ≰i,d ≰i and ad ≤i implies ad = 0. Since (a_1∨ c)d = a_1d ∨cd = cd ≠ 0.Now 0 ≠ (a_1 ∨ c)d = cd ≤ i, i is weakly prime, d ≰ i together implya_1 ∨ c ≤ i. So a_1≤i, a contradiction. Hence ab = 0.

The following result is proved by Calliap et.al.[9]
But this result is an outcome of the results proved above whoseproof is different.
Corollary 1) Let p and q be weakly prime elements of L whichare not prime then pq = 0.
Proof:- By Theorem (3.5), p, q ≤$\sqrt{0}$. Hence pq ≤p$\sqrt{0}$ = 0 (ByTheorem 3.5). Thus pq = 0.

Triple zeros of weakly 2-absorbing elements:

The concept of a triple zero of a weakly 2-absorbing ideal and free triple zero of weakly 2-absorbing ideal in a commutative ring is defined and studied by A. Badawi Certain Generalized Prime Elements et.al.[20]. The concept of a triple zero of a weakly 2-absorbing primary element is defined and studied by C.S.Manjarekar et.al.[54]. We extend the concept of a triple zero and free triple zero of a

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weakly 2-absorbing element in a compactly generated multiplicative lattices and obtain their properties.

Definition (3.9) Let i be a weakly 2-absorbing element of a multiplicative lattice L and a, b, c ∈ L. We say that (a,b,c) is a triple zero of i if abc = 0, ab≰i, bc≰i,ac≰i.

Definition (3.10) Let i be a weakly 2-absorbing element of a multiplicative lattice L and suppose $a_1a_2a_3$ ≤i for some elements a_1, a_2,a_3 ∈ L. We say that i is a free triple zero with respect to $a_1a_2a_3$ if (a,b,c) is not a triple zero of i for any a ≤a_1,

b ≤a_2,c ≤a_3.

Example 3.11) Let R = Z90. The set L = { i| i is an ideal of R} is a compactly generated multiplicative lattice. L = {0,< 1 >,< 2 >,< 3 >,< 5 >,< 6 >, < 9 >,< 10 >,< 15 >,< 18 >,< 30 >,< 45 >}. Then I =< 30 >2 L and 0 6=< 2 >< 3 >< 5 >_ I =< 30 > but < 2 >< 3 >*< 30 >, < 2 >< 5 >*< 30 >, < 3 >< 5 >*< 30 >= I. Hence iis not weakly 2-absorbing element of L.

Lemma 3.12) Let i be a weakly 2-absorbing element of L and suppose abd ≤ifor some elements a, b, d ∈ L such that (a,b,c) is not a triple zero of i for every c ≤ d. If ab≰i, then ad ≤i or bd≤i.

Proof:-Suppose ad ≰i or bd≰i. Then ad_1≰i and bd_2≰i for somed_1,d_2 ≤ d.Since (a, b,d_1) is not a triple zero of i and abd_1 ≤ i and ab≰i, ad_1≰i, we have bd_1 ≤i. Since (a, b,d_2) is not a triple zero of i and abd_2 ≤i and ab≰ i, bd_2≰i, we have ad_2 ≤i. Now since (a, b, (d_1 ∨d_2)) is not a triple zero of i and ab(d_1 ∨d_2) ≤i and ad ≰i, we have a(d_1 ∨d_2) ≤ i or b(d_1∨ d_2) ≤i. Suppose a(d_1 ∨d_2) = ad_1 ∨ ad_2 ≤i. Since ad_2 ≤i and ad_1 ≤i, we have a contradiction. Now suppose b(d_1 ∨d_2) = bd_1 ∨ bd_2 ≤i. Since bd_1≤ i and bd_2≤i, we have a contradiction. Hence ad ≤ i or bd ≤i.

Corollary 3.13) Let i be a weakly 2-absorbing element of L and suppose $a_1a_2a_3$ ≤i for some elements a_1,a_2, a_3 ∈ L such that i is a free triple zero with respect to $a_1a_2a_3$. Then if a ≤a_1, b ≤a_2, c ≤a_3, then ab ≤i or bc≤ i or ac ≤i.

Proof:- Since i is a free triple zero with respect to $a_1a_2a_3$. It follows that (a, b, c) is not a triple zero of i for every a ≤a_1, b ≤a_2, c ≤a_3. We have abc≤$a_1a_2a_3$ ≤i. Since (a, b, c) is not a triple zero of i we must have either ab ≤ i or bc ≤i or ac ≤i, ifabc = 0. If abc≠ 0 then 0 ≠abc ≤ i implies ab ≤ i or bc ≤ i or ac ≤i. Since iis weakly 2-absorbing element of L.

Theorem 3.14)i is weakly 2-absorbing element of L and 0 ≠ $a_1a_2a_3$≤i, a_1, a_2, a_3 ∈ L such that i is a free triple zero with respect to $a_1a_2a_3$. Then a_1a_2≤i or a_2a_3≤ i or a_1a_3≤i.

Suppose a_1a_2≰i, we claim that a_1a_3≤ i or a_2a_3 ≤i. Suppose a_1a_3≰ i or a_2a_3≰i. Then there exist q_1 ≤a_1 and q_2 ≤a_2 such that q_1a_3≰ i and q_2a_3≰i. Since qq_2a_3 ≤i and q_1a_3≰ i, q_2a_3≰i, we have q_1q_2 ≤i by lemma (3.12). Since a_1a_2≰iwe have ab≰i for some a ≤a_1, b ≤a_2. Since aba_3≤ i and ab≰i, we have aa_3≤i or ba_3 ≤ i by lemma (3.12).

Proof:-Case 1) Suppose aa_3≤ i but ba_3≰i. Since q_1ba_3≤i and ba_3≰iq_1a_3≰i and we have q_1b ≤ i by lemma (3.12). Since (a ∨q_1)ba_3 ≤iq_1a_3≰i we conclude that (a ∨q_1)a_3≰i. Since ba_3≰i and (a ∨q_1)a_3≰i we conclude that (a ∨q_1)b ≤i by lemma(3.12). Since (a ∨ q_1)b = ab ∨q_1b ≤i, so ab ≤i, a contradiction.

Case 2) Suppose ba_3 ≤ i but aa_3≰i. Since aq_2a_3≤i and aa_3≰ i, q_2a_3≰i we conclude that aq_2≤ i. Since a(b∨ q_2)a_3≤i and q_2a_3≰i we conclude (b ∨ q_2)a_3≰i.Since aa_3≰i, (b∨q_2)a_3≰ i, we conclude that a(b∨q_2) ≤ i by lemma (3.12). Since a(b ∨q_2) = ab ∨ aq_2 ≤ i, we have ab ≤i, a contradiction.

Case 3) Suppose aa_3 ≤ i and ba_3≤i. Since q_2a_3≰i, we conclude that (b∨q_2)a_3≰i. Since q_1(b ∨q_2)a_3≤i and q_1a_3≰i, (b ∨q_2)a_3≰i so q_1(b ∨q_2) = q_1b ∨q_1q_2≤ i by lemma (3.12). Since (q_1b∨q_1q_2) ≤ i we conclude bq_1≤i. As q_1a_3≰i, (a∨ q_1)a_3≰i.Since (a ∨q_1)q_2a_3≰ i and q_2a_3≰ i. (a ∨q_1)a_3≰i we have (a∨q_1)q_2 = aq_2 ∨q_1q_2 ≤i so aq_2 ≤ i. Now since (a ∨q_1)(b ∨q_2)a_3 ≤ i and (a ∨q_1)a_3≰ i and (b ∨q_2)a_3≰iwe have (a ∨q_1)(b ∨q_2) = ab ∨ aq_2 ∨ bq_1∨q_1q_2 ≤ i. By lemma (3.12) we concludethat ab ≤i, a contradiction. Hence a_1a_3 ≤ i or a_2a_3≤i.

REFERENCES

[1] D.D. Anderson, Abstract commutative ideal theory without chain condition, Algebra Universalis,6,(1976),131-145.

[2] R.P. Dilworth , Abstract Commutative Ideal theory, Pacific. J. Math.,12,(1962)481-498.

[3] F. Alarcon, D.D. Anderson, C. Jayaram, Some results on abstract commutative ideal theory, Periodica Mathemetica Hungerica, Vol 30 (1), (1995),pp.1-26.

[4] N. K. Thakre, C.S. Manjarekar and S. Maida, Abstract spectral theory II,Minimal characters and minimal spectrum of multiplicative lattices, Acta Sci.Math., 52 (1988) 53-67.

[5] Tessema, Belayneh, Venkateshwarlu, Certain Generalized Prime Ideals InBoolean like Semi-rings,International J. of Algebra, Hikari Ltd., Vol.8,(2014)No.14,663-669.

[6] U. Tekir,E. Yetkin,A. Badawi on 2 absorbing primary ideals in commutative rings, Bull korean Math. Soc.,51(2014),No.4,1163-1173.

[7] U. Tekir,E. Yetkin,C. Jayaram 2-absorbing and weakly 2-absorbing elements in multiplicatiove lattices, Communications in Algebra, 42,(2014),2338-2353.

[8] U. Tekir,E. Yetkin,F. Callialp On 2-absorbing primary and weakly 2-absorbingelements in multiplicative lattices,Italian Journal of Pure and Applied Mathematics, 34-2005, 263-276.

[9] C. Jayaram, U. Tekir, F.Callialp weakly prime elements in Multiplicative Lattices, Communications In Algebra 40, 2825-2840, 2012.