



On imbalances in multipartite multidigraphs

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Abstract

A k -partite r -digraph (multipartite multidigraph) (or briefly MMD) ($k \geq 3$, $r \geq 1$) is the result of assigning a direction to each edge of a k -partite multigraph that is without loops and contains at most r edges between any pair of vertices from distinct parts. Let $D(X_1, X_2, \dots, X_k)$ be a k -partite r -digraph with parts $X_i = \{x_{i1}, x_{i2}, \dots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d_{x_{ij}}^+$ and $d_{x_{ij}}^-$ be respectively the outdegree and indegree of a vertex x_{ij} in X_i . Define $a_{x_{ij}}$ (or simply a_{ij}) as $a_{ij} = d_{x_{ij}}^+ - d_{x_{ij}}^-$ as the imbalance of the vertex x_{ij} , $1 \leq j \leq n_i$. In this paper, we characterize the imbalances of k -partite r -digraphs and give a constructive and existence criteria for sequences of integers to be the imbalances of some k -partite r -digraph. Also, we show the existence of a k -partite r -digraph with the given imbalance set.

Keywords: digraph, outdegree, imbalance, maximum degree, oriented graph, multipartite multidigraph

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1. Introduction

A digraph without loops and without multi-arcs is called a simple digraph. The imbalance of a vertex v_i in a digraph as a_{v_i} (or simply a_i) = $d_{v_i}^+ - d_{v_i}^-$, where $d_{v_i}^+$ and $d_{v_i}^-$ are respectively the outdegree and indegree of v_i . The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. For undefined terms, we refer to [12].

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The following result [5] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1. *A sequence of integers $A = [a_1, a_2, \dots, a_n]$ with $a_1 \geq a_2 \geq \dots \geq a_n$ is an imbalance sequence if and only if it has sum zero and satisfies*

$$\sum_{i=1}^k a_i \leq k(n - k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 2. *A sequence of integers $A = [a_1, a_2, \dots, a_n]$ with $a_1 \leq a_2 \leq \dots \leq a_n$ is an imbalance sequence of a simple digraph if and only if for $1 \leq k < n$*

$$\sum_{i=1}^k a_i \geq k(k - n),$$

with equality when $k = n$.

Various results for imbalances in simple digraphs, bipartite and multipartite digraphs can be seen in [6, 8, 9, 27] while as characterization of imbalances in multidigraphs, bipartite multitournaments and tripartite multidigraphs can be found in [7, 10, 11].

An r -digraph ($r \geq 1$) is an orientation of a multigraph that is without loops and contains at most r edges between any pair of distinct vertices. Clearly 1-digraph is an oriented graph and complete 1-digraph is a tournament. Various results on scores and marks on tournaments and digraphs can be seen in [2, 13, 14, 15, 16, 17, 18, 24, 25] and on scores and degrees in hypertournaments can be found in [1, 3, 4, 19, 20, 21, 22, 23, 26]. The definition of an imbalance of a vertex in r -digraph is same as in a simple digraph. The following characterization for imbalances in multidigraphs can be found in Pirzada et al. [7].

Theorem 3. *A sequence $A = [a_1, a_2, \dots, a_p]$ of integers in non-decreasing order is the imbalance sequence of an r -digraph if and only if*

$$\sum_{i=1}^k a_i \geq rk(k - n),$$

for $1 \leq k \leq n$ with equality when $k = n$.

2. Imbalance sequences in multipartite multidigraphs

A k -partite r -digraph ($k \geq 3, r \geq 1$) (or briefly MMD) is the result of assigning a direction to each edge of a k -partite multidigraph which is without loops and having at most r edges between every pair of vertices, one vertex from each part. We note that k -partite 1-digraph is an oriented k -partite graph and complete k -partite 1-digraph is a k -partite tournament. The case when $k = 1$ gives multidigraph and when $k = 2$ gives bipartite multidigraph, and these have been studied in [9]. Thus throughout this paper the MMD under discussion will have three or more than three parts. Let $D(X_1, X_2, \dots, X_k)$ (or briefly D) be an MMD with parts $X_i = \{x_{i1}, x_{i2}, \dots, x_{in_i}\}$, $1 \leq i \leq k$. For any vertex x_{ij} in D , let $d_{x_{ij}}^+$ and $d_{x_{ij}}^-$ denote the outdegree and indegree. Define $a_{x_{ij}}$ (or simply a_{ij}) = $d_{x_{ij}}^+ - d_{x_{ij}}^-$ as the imbalance of the vertex x_{ij} , where $1 \leq j \leq n_i$. Note that we have $1 \leq i \leq k$ and $1 \leq j \leq n_i$, unless otherwise stated.

Let D be an MMD and $x_{ij} \in X_i, x_{mn} \in X_m$ and $x_{st} \in X_s$, where $i \neq m, m \neq s$ and $s \neq i$. If there are f arcs directed from vertex x_{ij} to x_{mn} and g arcs directed from x_{mn} to x_{ij} , with $0 \leq f \leq r, 0 \leq g \leq r$ and $0 \leq f + g \leq r$, it is denoted by $x_{ij}(f - g)x_{mn}$. An r -triple is an induced r -subdigraph of three vertices with one vertex from distinct part and is of the form $x_{ij}(f_1 - f_2)x_{mn}(g_1 - g_2)x_{st}(h_1 - h_2)x_{mn}$, where $0 \leq f_1, f_2, g_1, g_2, h_1, h_2, f_1 + f_2, g_1 + g_2, h_1 + h_2 \leq r$. Also an oriented triple (or 1-triple) in D is an induced 1-subdigraph of three vertices, with one vertex each from distinct part. An oriented triple is said to be transitive if it is of the form $x_{ij}(1 - 0)x_{mn}(1 - 0)x_{st}(0 - -1)x_{ij}$, or $x_{ij}(1 - 0)x_{mn}(0 - -1)x_{st}(0 - 0)x_{ij}$, or $x_{ij}(1 - 0)x_{mn}(0 - 0)x_{st}(0 - -1)x_{ij}$, or $x_{ij}(1 - 0)x_{mn}(0 - 0)x_{st}(0 - 0)x_{ij}$, or $x_{ij}(0 - -0)x_{mn}(0 - 0)x_{st}(0 - 0)x_{ij}$, otherwise it is intransitive. An MMD D is said to be transitive if each of its oriented triple is transitive. In particular, an r -triple C in an MMD D is transitive if every oriented triple of C is transitive.

We have the following observation.

Theorem 4. *Let D and D' be two MMD with the same imbalance sequences. Then D can be transformed to D' by successively transforming*

(i) *appropriate triples in one of the following ways.*

Either (a) by changing an intransitive oriented triple (cyclic triple) $x_{ij}(1 - 0)x_{mn}(1 - 0)x_{st}(1 - 0)x_{ij}$ to a transitive oriented triple $x_{ij}(0 - 0)x_{mn}(0 - 0)x_{st}(0 - 0)x_{ij}$ which has the same imbalance sequences, or vice versa,

or (b) by changing an intransitive oriented triple $x_{ij}(1 - 0)x_{mn}(1 - 0)x_{st}(0 - 0)x_{ij}$ to a transitive oriented triple $x_{ij}(0 - 0)x_{mn}(0 - 0)x_{st}(0 - -1)x_{ij}$ which has the same imbalance sequences, or vice versa.

or (ii) by changing the symmetric arcs $x_{ij}(1 - -1)x_{mn}$ to $x_{ij}(0 - 0)x_{mn}$, which has the same imbalance sequences, or vice versa.

Proof. This can be easily established. □

As a consequence of Theorem 3, we observe that among all MMD's having same imbalance sequences those with least number of arcs are transitive in nature.

Corollary 5. *Among all MMD's with given imbalance sequences, those with the fewest arcs are transitive.*

In an MMD, a vertex with indegree zero is called a transmitter. Evidently in a transitive oriented MMD with imbalance sequences $A_i = [a_{i1}, a_{i2}, \dots, a_{in_i}]$, any of the vertices with imbalances a_{in_i} , can act as a transmitter.

The next result provides a useful recursive test of checking whether the sequences of integers are the imbalance sequences of an MMD.

The following result is a combinatorial criterion for determining whether k sequences of integers are realizable as imbalances.

Theorem 6. *The k sequences of integers $A_i = [a_{i1}, a_{i2}, \dots, a_{in_i}]$, $1 \leq i \leq k$, in non-decreasing order are imbalance sequences of an MMD if and only if*

$$\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \dots + \sum_{j=1}^{m_k} a_{kj} \leq r \left[m_1 \prod_{j=1, j \neq 1}^k (n_j - m_j) + m_2 \prod_{j=1, j \neq 2}^k (n_j - m_j) + \dots + m_k \prod_{j=1, j \neq k}^k (n_j - m_j) \right] \tag{1}$$

or briefly

$$\sum_{i=1}^k \sum_{j=1}^{m_i} a_{ij} \leq r \left[\sum_{i=1}^k m_i \left(\prod_{j=1, j \neq i}^k (n_j - m_j) \right) \right],$$

for all sets of k integers m_i , $1 \leq m_i \leq n_i$, with equality when $m_i = n_i$, for all i .

Proof. The necessity of the condition follows from the fact that a directed multipartite multi sub-graph of an MMD induced by m_i vertices for all i , $1 \leq i \leq k$, $1 \leq m_i \leq n_i$ has a sum of imbalances zero and these vertices can gather at most

$$r \left[m_1 \prod_{j=1, j \neq 1}^k (n_j - m_j) + m_2 \prod_{j=1, j \neq 2}^k (n_j - m_j) + \dots + m_k \prod_{j=1, j \neq k}^k (n_j - m_j) \right]$$

imbalances from the remaining $n_i - m_i$ vertices.

For sufficiency, assume that $A_i = [a_{i1}, a_{i2}, \dots, a_{in_i}]$ be k sequences of integers in non-decreasing order satisfying conditions (1) but are not imbalance sequences of any MMD. Let these sequences be chosen in such a way that n_i , are the smallest possible and a_{11} is the least for the choice of n_i .

We consider the following two cases.

Case (i). Suppose equality in (1) holds for $m_j \leq n_j$, for $1 \leq j \leq k$ so that

$$\begin{aligned} & \sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj} \\ &= r \left[m_1 \prod_{j=1, j \neq 1}^k (n_j - m_j) + m_2 \prod_{j=1, j \neq 2}^k (n_j - m_j) + \cdots + m_k \prod_{j=1, j \neq k}^k (n_j - m_j) \right]. \end{aligned}$$

Consider the following k sequences.

$$\begin{aligned} A'_1 &= [a'_{1j}]_{j=1}^{m_1} = \left[a_{1j} - r \prod_{t=1, t \neq 1}^k (n_t - m_t) \right]_{j=1}^{m_1}, \\ A'_2 &= [a'_{2j}]_{j=1}^{m_2} = \left[a_{2j} - r \prod_{t=1, t \neq 2}^k (n_t - m_t) \right]_{j=1}^{m_2}, \\ &\vdots \\ A'_k &= [a'_{kj}]_{j=1}^{m_k} = \left[a_{kj} - r \prod_{t=1, t \neq k}^k (n_t - m_t) \right]_{j=1}^{m_k} \end{aligned}$$

For $1 \leq r_i < m_i$ for all $1 \leq i \leq k$, we have

$$\begin{aligned} & \sum_{i=1}^k \sum_{j=1}^{r_i} a'_{ij} = \sum_{j=1}^{r_1} a'_{1j} + \sum_{j=1}^{r_2} a'_{2j} + \cdots + \sum_{j=1}^{r_k} a'_{kj} \\ &= \sum_{j=1}^{r_1} \left(a_{1j} - r \prod_{t=1, t \neq 1}^k (n_t - m_t) \right) + \sum_{j=1}^{r_2} \left(a_{2j} - r \prod_{t=1, t \neq 2}^k (n_t - m_t) \right) \\ &+ \cdots + \sum_{j=1}^{r_k} \left(a_{kj} - r \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\ &= \left(\sum_{j=1}^{r_1} a_{1j} + \sum_{j=1}^{r_2} a_{2j} + \cdots + \sum_{j=1}^{r_k} a_{kj} \right) \\ &- r \left(r_1 \prod_{t=1, t \neq 1}^k (n_t - m_t) + r_2 \prod_{t=1, t \neq 2}^k (n_t - m_t) + \cdots + r_k \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \end{aligned}$$

$$\begin{aligned}
 &\leq r \left(r_1 \prod_{j=1, j \neq 1}^k (n_j - r_j) + r_2 \prod_{j=1, j \neq 2}^k (n_j - r_j) + \cdots + r_k \prod_{j=1, j \neq k}^k (n_j - r_j) \right) \\
 &- r \left(r_1 \prod_{t=1, t \neq 1}^k (n_t - m_t) + r_2 \prod_{t=1, t \neq 2}^k (n_t - m_t) + \cdots + r_k \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\
 &\leq r \left(r_1 \prod_{t=1, t \neq 1}^k (m_t - r_t) + r_2 \prod_{t=1, t \neq 2}^k (m_t - r_t) + \cdots + r_k \prod_{t=1, t \neq k}^k (m_t - r_t) \right) \\
 &= r \sum_{i=1}^k r_i \left(\prod_{t=1, t \neq i}^k (m_t - r_t) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{i=1}^k \sum_{j=1}^{m_i} a'_{ij} &= \sum_{j=1}^{m_1} a'_{1j} + \sum_{j=1}^{m_2} a'_{2j} + \cdots + \sum_{j=1}^{m_k} a'_{kj} \\
 &= \sum_{j=1}^{m_1} \left(a_{1j} - r \prod_{t=1, t \neq 1}^k (n_t - m_t) \right) + \sum_{j=1}^{m_2} \left(a_{2j} - r \prod_{t=1, t \neq 2}^k (n_t - m_t) \right) \\
 &+ \cdots + \sum_{j=1}^{m_k} \left(a_{kj} - r \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\
 &= \left(\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj} \right) \\
 &- r \left(m_1 \prod_{t=1, t \neq 1}^k (n_t - m_t) + m_2 \prod_{t=1, t \neq 2}^k (n_t - m_t) + \cdots + m_k \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\
 &= r \left(m_1 \prod_{j=1, j \neq 1}^k (n_j - m_j) + m_2 \prod_{j=1, j \neq 2}^k (n_j - m_j) + \cdots + m_k \prod_{j=1, j \neq k}^k (n_j - m_j) \right) \\
 &- r \left(m_1 \prod_{t=1, t \neq 1}^k (n_t - m_t) + m_2 \prod_{t=1, t \neq 2}^k (n_t - m_t) + \cdots + m_k \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\
 &= r \sum_{i=1}^k m_i \left(\prod_{j=1, j \neq i}^k (n_j - m_j) \right) - r \sum_{i=1}^k m_i \left(\prod_{t=1, t \neq i}^k (n_t - m_t) \right) \\
 &= 0.
 \end{aligned}$$

Thus the k sequences $A'_i = [a'_{ij}]_{j=1}^{m_i}$, for all $1 \leq i \leq k$ satisfy (1) and by the minimality of n_i the sequences A'_i are the imbalance sequences of some MMD $D'(V'_1, V'_2, \dots, V'_k, E')$.

Now consider the following k sequences.

$$\begin{aligned}
 A''_1 &= \left[a_{1(m_1+j)} + r \prod_{t=1, t \neq 1}^k m_t \right]_{j=1}^{n_1} \\
 A''_2 &= \left[a_{2(m_2+j)} + r \prod_{t=1, t \neq 2}^k m_t \right]_{j=1}^{m_2}, \\
 &\vdots \\
 A''_k &= \left[a_{k(m_k+j)} + r \prod_{t=1, t \neq k}^k m_t \right]_{j=1}^{m_k}
 \end{aligned}$$

For all $1 \leq i \leq k$ and $1 \leq r_i \leq n_i - m_i$, we have

$$\begin{aligned}
 \sum_{i=1}^k \sum_{j=1}^{r_i} a''_{ij} &= \sum_{j=1}^{r_1} a''_{1j} + \sum_{j=1}^{r_2} a''_{2j} + \cdots + \sum_{j=1}^{r_k} a''_{kj} \\
 &= \sum_{j=1}^{r_1} \left(a_{1(m_1+j)} + r \prod_{t=1, t \neq 1}^k m_t \right) + \sum_{j=1}^{r_2} \left(a_{2(m_2+j)} + r \prod_{t=1, t \neq 2}^k m_t \right) \\
 &+ \cdots + \sum_{j=1}^{r_k} \left(a_{k(m_k+j)} + r \prod_{t=1, t \neq k}^k m_t \right) \\
 &= \sum_{j=1}^{r_1} \left(a_{1(m_1+j)} + r \prod_{t=1, t \neq 1}^k m_t \right) + \sum_{j=1}^{r_2} \left(a_{2(m_2+j)} + r \prod_{t=1, t \neq 2}^k m_t \right) \\
 &+ \cdots + \sum_{j=1}^{r_k} \left(a_{k(m_k+j)} + r \prod_{t=1, t \neq k}^k m_t \right) \\
 &= \left(\sum_{j=1}^{m_1+r_1} a_{1j} + \sum_{j=1}^{m_2+r_2} a_{2j} + \cdots + \sum_{j=1}^{m_k+r_k} a_{kj} \right) \\
 &- \left(\sum_{j=1}^{m_1} a_{1j} + \sum_{j=1}^{m_2} a_{2j} + \cdots + \sum_{j=1}^{m_k} a_{kj} \right) \\
 &+ r \left(r_1 \prod_{t=1, t \neq 1}^k m_t + r_2 \prod_{t=1, t \neq 2}^k m_t + \cdots + r_k \prod_{t=1, t \neq k}^k m_t \right)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \sum_{i=1}^k \sum_{j=1}^{r_i} a''_{ij} \\
 & \leq r(m_1 + r_1) \prod_{j=1, j \neq 1}^k (n_t - (m_t + r_t)) + r(m_2 + r_2) \prod_{j=1, j \neq 2}^k (n_t - (m_t + r_t)) \\
 & + \cdots + r(m_k + r_k) \prod_{j=1, j \neq k}^k (n_t - (m_t + r_t)) \\
 & - r \left(m_1 \prod_{t=1, t \neq 1}^k (n_t - m_t) + m_2 \prod_{t=1, t \neq 2}^k (n_t - m_t) + \cdots + m_k \prod_{t=1, t \neq k}^k (n_t - m_t) \right) \\
 & + r \left(r_1 \prod_{t=1, t \neq 1}^k m_t + r_2 \prod_{t=1, t \neq 2}^k m_t + \cdots + r_k \prod_{t=1, t \neq k}^k m_t \right) \\
 & \leq r \left(r_1 \prod_{t=1, t \neq 1}^k (n_t - m_t - r_t) + r_2 \prod_{t=1, t \neq 2}^k (n_t - m_t - r_t) + \cdots + r_k \prod_{t=1, t \neq k}^k (n_t - m_t - r_t) \right) \\
 & = r \sum_{i=1}^k r_i \left(\prod_{t=1, t \neq i}^k (n_t - m_t - r_t) \right)
 \end{aligned}$$

with equality when $r_i = n_i - m_i$ for all $i, 1 \leq i \leq k$. Therefore by the minimality of n_i , the k sequences A''_i form the imbalance sequences of some MMD $D''(V''_1, V''_2, \dots, V''_k, E'')$.

Construct an MMD $D(V_1, V_2, \dots, V_k, E)$ as follows.

Let $V_1 = V'_1 \cup V''_1, V_2 = V'_2 \cup V''_2, \dots, V_k = V'_k \cup V''_k$ and $V'_i \cap V''_j = \phi$ for all i, j . The arc set E contains (a) all the arcs between V'_i and $V'_j, i \neq j$, (b) all arcs between V''_i and $V''_j, i \neq j$ and (c) r arcs from each vertex of V'_i to every vertex of V''_j .

Then $D(V_1, V_2, \dots, V_k)$ has imbalance sequences A_i , which is a contradiction. Clearly this contradicts the given assumption.

Case (ii). Assume that the strict inequality holds in (1) for $m_i \neq n_i$. Let $A_1 = [a_{11} - 1, a_{12}, \dots, a_{1(n_1-1)}, a_{1n_1} + 1]$ and Let $A_j = [a_{j1}, a_{j2}, \dots, a_{jn_j}]$ for all $j, 2 \leq j \leq k$, so that A_i satisfy the conditions (1). Therefore by the minimality of a_{11} , the sequences A_i are the imbalance sequences of some MMD $D_1(Y_1, Y_2, \dots, Y_k)$. Let $a_{x_{11}} = a_{11} - 1$ and $a_{x_{1n_1}} = a_{1n_1} + 1$. Since $a_{x_{1n_1}} > a_{x_{11}} + 1$, there exists a vertex x_{jp} in $Y_j, 2 \leq j \leq k, 1 \leq p \leq n_j$, such that $x_{1n_1}(0 - -0)x_{jp}(1 - -0)x_{11}$, or $x_{1n_1}(1 - -0)x_{jp}(0 - -0)x_{11}$, or $x_{1n_1}(1 - -0)x_{jp}(1 - -0)x_{11}$, or $x_{1n_1}(0 - -0)x_{jp}(0 - -0)x_{11}$ in $D_1(Y_1, Y_2, \dots, Y_k)$, and if these are changed to $x_{1n_1}(0 - -1)x_{jp}(0 - -0)x_{11}$, or $x_{1n_1}(0 - -0)x_{jp}(0 - -1)x_{11}$, or $x_{1n_1}(0 - -0)x_{jp}(0 - -0)x_{11}$, or $x_{1n_1}(0 - -1)x_{jp}(0 - -1)x_{11}$ respectively, the result is an MMD with imbalance sequences A_i , which is again a contradiction. This completes the proof. \square

3. Imbalance sets in multipartite multidigraphs

The set of distinct imbalances of the vertices in an MMD is called its imbalance set. Now we give the existence of an MMD with a given imbalance set.

Theorem 8. *Let $P = \{p_1, p_2, \dots, p_m\}$ and $Q = \{-q_1, -q_2, \dots, -q_n\}$, where $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n$ are positive integers with $p_1 < p_2 < \dots < p_m$ and $q_1 < q_2 < \dots < q_n$ and the greatest common divisor $(p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n) = t$, where $1 \leq t \leq r$. Then there exists an MMD with imbalance set $P \cup Q$.*

Proof. Since $(p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_n) = t$ there exist positive integers $f_1, f_2, \dots, f_m, g_1, g_2, \dots, g_n$ with $f_1 < f_2 < \dots < f_m$ and $g_1 < g_2 < \dots < g_n$ such that $p_i = tf_i$ for $1 \leq i \leq m$ and $q_j = tg_j$ for $1 \leq j \leq n$. First assume that k is even, say $k = 2s$ where $s \geq 1$. Construct an MMD $D(X_1, X_2, \dots, X_k)$ as follows. Let

$$\begin{aligned} X_1 &= X_{11} \cup X_{12} \cup \dots \cup X_{1m} \cup X_1^1 \cup X_1^2 \cup \dots \cup X_1^n, \\ X_2 &= X_{21} \cup X_{22} \cup \dots \cup X_{2m} \cup X_2^1 \cup X_2^2 \cup \dots \cup X_2^n, \\ X_3 &= X_{31} \cup X_{32} \cup \dots \cup X_{3m} \cup X_3^1 \cup X_3^2 \cup \dots \cup X_3^n, \\ &\dots \\ X_{2s} &= X_{2s1} \cup X_{2s2} \cup \dots \cup X_{2sm} \cup X_{2s}^1 \cup X_{2s}^2 \cup \dots \cup X_{2s}^n, \end{aligned}$$

with $X_{ij} \cap X_{uv} = \emptyset, X_{ij} \cap X_u^v = \emptyset, X_i^j \cap X_u^v = \emptyset, |X_{1i}| = g_1$ for all $i, 1 \leq i \leq m, |X_1^i| = g_i$ for all $i, 1 \leq i \leq n, |X_{2i}| = f_i$ for all $i, 1 \leq i \leq m, |X_2^i| = f_1$ for all $i, 1 \leq i \leq n, |X_{i1}| = g_1$ for all odd $i, 3 \leq i \leq 2s - 1$, and $|X_{i1}| = f_1$ for all even $i, 4 \leq i \leq 2s$. Let there be t arcs directed from each vertex of X_{1i} to every vertex of X_{2i} for all $1 \leq i \leq m; t$ arcs directed from each vertex of X_1^i to every vertex of X_2^i $1 \leq i \leq n$ and t arcs directed from each vertex of X_{i1} to every vertex of $X_{(i+1)1}$ for all odd $i, 3 \leq i \leq 2s - 1$ so that we obtain $2s$ -partite r -digraph MMD with imbalance of vertices as follows.

For $1 \leq i \leq m,$

$$a_{x_{1i}} = t|X_{2i}| - 0 = tf_i = p_i, \text{ for all } x_{1i} \in X_{1i};$$

for $1 \leq j \leq n,$

$$a_{x_1^j} = t|X_2^j| - 0 = tf_1 = p_1, \text{ for all } x_1^j \in X_1^j;$$

for $1 \leq j \leq m,$

$$a_{x_{2i}} = 0 - t|X_{1i}| = -tg_1 = -q_1, \text{ for all } x_{2i} \in X_{2i};$$

for $1 \leq j \leq n,$

$$a_{x_2^j} = 0 - t|X_{11}| = -tg_1 = -q_1, \text{ for all } x_2^j \in X_2^j;$$

for all odd $i, 3 \leq i \leq 2s - 1$

$$a_{x_{i1}} = t|X_{(i+1)1}| - 0 = tf_1 = p_1, \text{ for all } x_{i1} \in X_{i1};$$

and for even i , $4 \leq i \leq 2s$,

$$a_{x_{i1}} = 0 - t|X_{(i-1)1}| = -tg_1 = -q_1, \text{ for all } x_{i1} \in X_{i1}.$$

Therefore imbalance set of $D(X_1, X_2, \dots, X_k)$ is $P \cup Q$.

By using the same argument as above, it can be shown that the result holds for odd k also. Hence the proof is complete. \square

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