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New bounds on the hyper-Zagreb index for the simple connected graphs

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Abstract

The hyper-Zagreb index of a simple connected graph G is defined by

$$\chi^2(G) = \sum_{uv \in E(G)} (d(u) + d(v))^2.$$

In this paper, we establish, analyze and compare some new upper bounds on the Hyper-Zagreb index in terms of the number of vertices (n), number of edges (m), maximum vertex degree (Δ), and minimum vertex degree (δ), first Zagreb index $M_1(G)$, second Zagreb index $M_2(G)$, harmonic index $H(G)$, and inverse edge degree $IED(G)$. In addition, we give the identities on Hyper-Zagreb index and its coindex for the simple connected graphs.

Keywords: hyper-Zagreb index, Zagreb indices, forgotten index
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1. Introduction

Mathematical chemistry is a branch of theoretical chemistry using mathematical methods to discuss and predict molecular properties without necessarily referring to quantum mechanics.

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Chemical graph theory is a branch of mathematical chemistry which applies graph theory in mathematical modeling of chemical phenomena. Topological indices are the numerical values associated with chemical structures which are used to study and predict the physicochemical property correlations of organic compounds in QSAR and QSPR studies [9, 15].

A molecular graph is a representation of the structural formula of a chemical compound such that its vertices correspond to the atoms and the edges to the bonds. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G), N_G(u)$ denotes the (first) neighbors of u in G and the degree of u is denoted by $d(u) = d_G(u) = |N_G(u)|$ and the degree of edge $e = uv$ is denoted by $d(e) = d(u) + d(v) - 2$.

One of the most popular and most used molecular descriptors are the Novel *first and second Zagreb indices*, introduced by Gutman and Trinajstić [12] and are defined as

$$M_1(G) = \sum_{v \in V(G)} d(v)^2 \text{ and } M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

Numerous papers were recorded in the literature regarding the mathematical and chemical properties on Zagreb indices and the surveys on Zagreb indices. For the recent outcomes on Zagreb indices, see [8, 17, 3] and references therein. Li and Zheng [14] introduced the generalized version of the first Zagreb index. For $\alpha \in \mathbb{R}$ and G be any graph satisfies the important identity (1):

$$M_1^{\alpha+1}(G) = \sum_{v \in V(G)} d(v)^{\alpha+1} = \sum_{uv \in E(G)} [d(u)^\alpha + d(v)^\alpha]. \tag{1}$$

Very recently, Furtula and Gutman [11, 12] re-introduced the *forgotten topological index*, defined by

$$F(G) = \sum_{v \in V(G)} d(v)^3.$$

Ashrafi, Došlić and Hamzeha introduced the concept of sum of nonadjacent vertex degree pairs of the graph G, known as *first and second Zagreb coindices* [1] and are defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d(u) + d(v)] \text{ and } \overline{M}_2(G) = \sum_{uv \notin E(G)} d(u)d(v).$$

2. Basic notions and Preliminaries

The degree of a vertex is denoted by $d(v_i)$ for $i = 1, 2, \ldots, n$ such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. For any $v \in V(G)$ is said to be pendant vertex, if $d(v) = 1$ and p denotes the number of pendent vertices of G. Let $\Delta = \Delta(G), \delta_1 = \delta_1(G)$ and $\delta = \delta(G)$ denotes the maximum, minimum non pendant and minimum vertex degree of G respectively. Similarly, for $e = uv \in E(G)$, then the degree of a edge is denoted by $d(e) = d(u) + d(v) - 2$. Let $\overline{G}$ denotes the complement of G, with the same vertex set such that two vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The degree of the same vertex in $\overline{G}$ is then given by $n - 1 - d(v)$ and

$\overline{m} = \frac{n(n-1)}{2} - m$. The *line graph* $L(G)$ obtained from G in which $V(L(G)) = E(G)$, where two vertices of $L(G)$ are adjacent if and only if they are adjacent edges of G.

As usual P_n, $K_{1,n-1}$, C_n, K_n denote *path, star, cycle and the complete graph* on n vertices, respectively. The graph $K^*_{2,n-2}$ is a connected graph with n vertices obtained from the complete bipartite graph $K_{2,n-2}$ with two vertices of degree $n-2$ are joined by a new edge. The *wheel graph* W_n is obtained by connecting a K_1 to all vertices of a cycle C_{n-1}. The *helm graph* H_n is obtained from W_n by adjoining a pendant edge at each vertex of the cycle. The flower Fl_n is obtained from the helm H_n by joining each pendent vertex to the central vertex of the helm. A graph G is called bidegreed if its vertex degree is either Δ or δ with $\Delta > \delta \geq 1$. Let $B_{n,t}$ be the graph on n vertices with exactly t vertices of degree $n-1$ and the remaining of $n-t$ vertices forming an independent set.

3. Main Results

In 2010, Zhou and Trinajstić introduced the general sum-connectivity index [22]:

$$\chi^\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha.$$

Obviously, $\chi^0(G) = m, \chi^1(G) = M_1(G)$. In 2013, Shirdel, Rezapour and Sayadi [19] defined the *Hyper-Zagreb index,* which is simply the general sum-connectivity index with $\alpha = 2$ in [22]. This paper deals with the mathematical properties of the general sum-connectivity index for $\alpha = 2$.

By the definition of Hyper-Zagreb index, we have that $\chi^2(G) = F(G) + 2M_2(G)$. Clearly, the mathematical properties of $\chi^2(G)$ partially depends on the forgotten index and the second Zagreb index. So the bounds for these indices automatically leads to the bounds for $\chi^2(G)$.

In 2006, Cioabă [4] established the upper bound for $F(G)$ in terms of n, m, δ, Δ and $M_1(G)$.

Theorem 3.1. (See [4]) *Let G be a connected graph. Then*

$$F(G) \leq \frac{2m - (\Delta^2 - \delta^2)}{n} M_1(G) + \frac{2m(n-1)(\Delta^2 - \delta^2)}{n}$$

with equality holds if and only if G is regular or $G = B_{n,t}$ for some t with $1 \leq t \leq n$.

In 2010, Zhou and Trinajstić [23] present the upper bound for $F(G)$ involving the first and second Zagreb indices.

Theorem 3.2. (See [23]) *Let G be a graph on n vertices and $m \geq 1$ edges. Then*

$$F(G) \leq 2M_2(G) + nM_1(G) - 4m^2 \tag{2}$$

with equality if and only if any two non-adjacent vertices have equal degrees.

Our next intention is to give upper bounds on $F(G)$ only in terms of n, m, Δ, δ.

Theorem 3.3. *Let G be a simple graph with $n(\geq 3)$ vertices and m edges. Then*

$$F(G) \leq \Delta^3 + \Delta - (m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right] \quad (3)$$

with equality holds if and only if G is $K_{1,n-1}$ or K_n or $K^*_{2,n-2}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Choose a vertex whose degree is Δ in $V(G)$ and label it as v_1. Then

$$F(G) = d_G(v_1)^3 + \sum_{i=2}^{n} d_G(v_i)^3.$$

Let G_1 be a subgraph obtained by deleting the vertex v_1 in G. So G_1 has $m - \Delta$ edges and $\{v_2, v_3, \ldots, v_n\}$ vertices.

$$\begin{aligned}
F(G) &= \Delta^3 + \sum_{v_i \in N(v_1)} d_G(v_i)^3 + \sum_{v_i \in V(G)\setminus\{N(v_1)\cup v_1\}} d_G(v_i)^3 \\
&= \Delta^3 + \sum_{v_i \in N(v_1)} (d_{G_1}(v_i)+1)^3 + \sum_{v_i \in V(G_1)\setminus N(v_1)} d_{G_1}(v_i)^3 \\
&= \Delta^3 + \Delta + 6(m-\Delta) + 3\sum_{v_i \in N(v_1)} d_{G_1}(v_i)^2 + \sum_{v_i \in N(v_1)} d_{G_1}(v_i)^3 + \sum_{v_i \in V(G_1)\setminus N(v_1)} d_{G_1}(v_i)^3 \\
&\leq \Delta^3 + \Delta + 6(m-\Delta) + 3\sum_{v_i \in V(G_1)} d_{G_1}(v_i)^2 + \sum_{v_i \in V(G_1)} d_{G_1}(v_i)^3.
\end{aligned}$$

Taking into the consideration of the upper bounds of $M_1(G)$ and $M_2(G)$ from [5, 6] and [7] respectively for the graph G_1, we have

$$\sum_{v_i \in V(G_1)} d_{G_1}(v_i)^2 \leq \left(\frac{2(m-\Delta)}{n-2} + n - 3\right),$$

$$M_2(G_1) \leq 2(m-\Delta)^2 - (n-2)(m-\Delta)(\delta-1) + \frac{1}{2}(\delta-2)(m-\Delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right).$$

Since v_1 is chosen with $d_G(v_1) = \Delta$. Then G may have at most k-vertices of degree δ $(2 \leq k \leq \Delta)$ with $d(v_1, v_k) = 1$, for some k. If there exists at least one such v_k, then we have $d_{G_1}(v_k) = \delta - 1$.

By utilizing (2) for the graph G_1 and using the above bounds, we have

$$\begin{aligned}
\sum_{v_i \in V(G_1)} d_{G_1}(v_i)^3 &\leq (n-1)\sum_{v_i \in V(G_1)} d_{G_1}(v_i)^2 + 2M_2(G_1) - 4(m-\Delta)^2 \\
&\leq (n-1)\sum_{v_i \in V(G_1)} d_{G_1}(v_i)^2 - 2(m-\Delta)(n-2)(\delta-1) \\
&\quad + (m-\Delta)(\delta-2)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right).
\end{aligned}$$

Thus, we get

$$F(G) \leq \Delta^3 + \Delta - (m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right].$$

First part of the proof is done. Now suppose, $G \in K^*_{2,n-2}$. Then $m = 2n-3$ and $\Delta = n-1$ and we get

$$\begin{aligned}\Delta^3 + \Delta - &(m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right]\\ &= (n-1)^3 + (n-1) - (n-2)\left[2(n-5) - (n+2)(n-1)\right]\\ &= 2(n-1)^3 + (n-2)2^3\\ &= F\left(K^*_{2,n-2}\right).\end{aligned}$$

If $G \in K_{1,n-1}$, with $m = n-1$ and $\Delta = n-1$, we get

$$\begin{aligned}\Delta^3 + \Delta - &(m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right]\\ &= (n-1)^3 + (n-1)\\ &= F\left(K_{1,n-1}\right).\end{aligned}$$

If $G \in K_n$, with $m = \frac{n(n-1)}{2}$ and $\Delta = n-1$, we get

$$\begin{aligned}\Delta^3 + \Delta - &(m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right]\\ &= n(n-1)^2\\ &= F\left(K_n\right).\end{aligned}$$

Similarly we get the converse part, which completes the proof. □

Suppose, if $d(v_1, v_k) \neq 1$ for any $k(2 \leq k \leq n)$, then $d_{G_1}(v_k) = \delta$. By replacing $\delta - 1$ by δ in (3) immediately leads to the following inequality.

$$F(G) \leq \Delta^3 + \Delta - (m-\Delta)\left[2\left[(n-2)\delta-3\right] - (n+\delta+1)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right]. \tag{4}$$

The inequality (4) fails only for the class of $K^*_{2,n-2}$ graphs.

By the above theorem, we can state the following two corollaries.

Corollary 3.1. *Let G be a simple graph with $n(\geq 3)$ vertices and m edges. Then*

$$\begin{aligned}\chi^2(G) \leq\, & 2M_2(G) + \Delta^3 + \Delta\\ & - (m-\Delta)\left[2\left[(n-2)(\delta-1)-3\right] - (n+\delta)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right]\end{aligned} \tag{5}$$

with equality holds if and only if G is $K_{1,n-1}$ or K_n or $K^*_{2,n-2}$.

Corollary 3.2. *Let G be a $K^*_{2,n-2}$ free simple graph with $n(\geq 3)$ vertices and m edges. Then*

$$\chi^2(G) \leq 2M_2(G) + \Delta^3 + \Delta - (m-\Delta)\left[2\left[(n-2)\delta - 3\right] - (n+\delta+1)\left(\frac{2(m-\Delta)}{n-2} + n - 3\right)\right] \tag{6}$$

with equality holds if and only if G is $K_{1,n-1}$ or K_n.

Theorems 3.1 and 3.2 immediately leads to the following corollaries for $\chi^2(G)$.

Corollary 3.3. *Let G be a connected graph. Then*

$$\chi^2(G) \leq 2M_2(G) + \frac{2m - (\Delta^2 - \delta^2)}{n} M_1(G) + \frac{2m(n-1)(\Delta^2 - \delta^2)}{n} \tag{7}$$

with equality holds if and only if G is regular or $G = B_{n,t}$ for some t with $1 \leq t \leq n$.

Corollary 3.4. *Let G be a graph on n vertices and $m \geq 1$ edges. Then*

$$\chi^2(G) \leq 4M_2(G) + nM_1(G) - 4m^2 \tag{8}$$

with equality if and only if any two non-adjacent vertices have equal degrees.

Note that since the bounds in (5) and (6) are obtained using the bound in (2) partially. So (8) is always better than (5) and (6). But, (5) and (6) are incomparable with (7). For the flower graphs F_n with $n \geq 4$, the bound in (6) is finer than (7) and for $L(F_n)$ the bound in (7) is finer than (6).

Next, we are ready to give some new upper bounds for $\chi^2(G)$ involving the other vertex and edge based topological indices.

Theorem 3.4. *Let G be any simple graph with n vertices and m edges. Then*

$$\chi^2(G) \leq 2(\Delta + \delta)M_1(G) - 4m\Delta\delta \tag{9}$$

with equality if and only if G is regular.

Proof. The vertex degree $d(v_i)$ is bounded by $\delta \leq d(v_i) \leq \Delta$ for $i = 1, 2, \ldots, n$. In analogy the edge degree is bounded by $2(\delta - 1) \leq d(e_i) \leq 2(\Delta - 1)$ for $i = 1, 2, \cdots, m$. The edges of G are labeled as $e_1, e_2, \ldots, e_m$ such that $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$

$$\begin{aligned}
\sum_{i=1}^{m} d(e_i)^2 &= \sum_{i=1}^{m} \left[d(e_i)\left(d(e_i) - d(e_m)\right) + d(e_i)d(e_m)\right] \\
&\leq \sum_{i=1}^{m} \left[d(e_1)\left(d(e_i) - d(e_m)\right) + d(e_i)d(e_m)\right] \\
&\leq \sum_{i=1}^{m} \left[2(\Delta - 1)\left(d(e_i) - 2(\delta - 1)\right) + d(e_i)2(\delta - 1)\right] \\
&= (2(\Delta + \delta) - 4)\sum_{i=1}^{m} d(e_i) - 4(\Delta - 1)(\delta - 1)\sum_{i=1}^{m} 1.
\end{aligned}$$

$$\sum_{uv\in E(G)} [d(u)+d(v)-2]^2 \leq (2(\Delta+\delta)-4) \sum_{uv\in E(G)} [d(u)+d(v)-2] - 4(\Delta-1)(\delta-1) \sum_{uv\in E(G)} 1.$$

Finally, by (1), we have

$$\chi^2(G) \leq 2(\Delta+\delta) \sum_{uv\in E(G)} [d(u)+d(v)] + 4m - 4\left(\Delta+\delta\right)m - 4m(\Delta-1)(\delta-1).$$

The equality holds if and only if G is regular. This completes the proof. □

Now, we improve (9). Next we contemplate on the Harmonic index, which is the another variant Randić index. The Harmonic index $H(G)$ was first emerged in the conjectures of the computer program Graffiti [10]:

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}.$$

Theorem 3.5. *Let G be any simple graph with n vertices and m edges. Then*

$$\chi^2(G) \leq (2\Delta+2\delta+1)M_1(G) + 2\Delta\delta H(G) - 2m(2\Delta\delta+\Delta+\delta) \tag{10}$$

with equality if and only if G is regular.

Proof. Followed by the same argument as in the proof of the Theorem 3.4, we have

$$\sum_{uv\in E(G)} [d(u)+d(v)]^2 \leq \sum_{uv\in E(G)} (2\Delta+2\delta)\left[d(u)+d(v)\right] - \sum_{uv\in E(G)} (2\Delta)(2\delta)$$

For any edge $uv \in E(G)$, it is true that $\dfrac{1}{d(u)+d(v)} < 1$ and using in the above inequality, we have

$$\sum_{uv\in E(G)} \left[1-\frac{1}{d(u)+d(v)}\right] [d(u)+d(v)]^2 \leq 2(\Delta+\delta) \sum_{uv\in E(G)} \left[1-\frac{1}{d(u)+d(v)}\right] [d(u)+d(v)] - 4\Delta\delta \sum_{uv\in E(G)} \left[1-\frac{1}{d(u)+d(v)}\right]$$

$$\sum_{uv\in E(G)} [d(u)+d(v)]^2 - \sum_{uv\in E(G)} [d(u)+d(v)] \leq 2(\Delta+\delta) \sum_{uv\in E(G)} [d(u)+d(v)] - 2(\Delta+\delta) \sum_{uv\in E(G)} 1 - 4\Delta\delta \sum_{uv\in E(G)} 1 - \sum_{uv\in E(G)} \frac{4\Delta\delta}{d(u)+d(v)}$$

after expanding, we get

$$\chi^2(G) - M_1(G) \leq 2(\Delta+\delta)M_1(G) - 2(\Delta+\delta)m - 4m\Delta\delta + 2\Delta\delta H(G).$$

The equality holds for the regular graphs and this completes the proof. □

Naurmi [18] introduced the inverse degree and also it was attracted attention through conjectures of the computer program Graffiti [10]. The inverse degree of a graph G with no isolated vertices are defined as

$$ID(G) = \sum_{u \in V(G)} \frac{1}{d(u)}.$$

For the recent results of the inverse degree, see [2, 21]. In analogy, we now define the *inverse edge degree* of a graph G with $n(>2)$ vertices and with non isolated edges are defined as

$$IED(G) = \sum_{e \in E(G)} \frac{1}{d(e)}.$$

Theorem 3.6. *Let G be any simple graph with $n(>2)$ vertices and with non isolated edges. Then*

$$\chi^2(G) \leq (2\Delta+2\delta+1)M_1(G) + 4(\Delta-1)(\delta-1)IED(G) - 2m(2\Delta\delta+\Delta+\delta-1) \quad (11)$$

with equality if and only if G is regular.

Proof. Since G has no isolated edges, then $d(u)+d(v)>2$ and with the assumptions of the proof of Theorem 3.4, we have

$$\sum_{i=1}^{m}\left[1-\frac{1}{d(e_i)}\right]d(e_i)^2 \leq 2(\Delta+\delta-2)\sum_{i=1}^{m}\left[1-\frac{1}{d(e_i)}\right]d(e_i) - 4(\Delta-1)(\delta-1)\sum_{i=1}^{m}\left[1-\frac{1}{d(e_i)}\right]$$

after expanding, we get

$$\sum_{uv \in E(G)}[d(u)+d(v)]^2 \leq (2\Delta+2\delta+1)\sum_{uv \in E(G)}[d(u)+d(v)] + 6m - 4m(\Delta-1)(\delta-1)$$
$$- 4m(\Delta+\delta) + 4(\Delta-1)(\delta-1)\sum_{uv \in E(G)}\frac{1}{d(u)+d(v)-2}.$$

The equality holds for the regular graphs and this completes the proof. □

Remark 3.1. Let $x, y \in \mathbb{N}$, we have $1-\frac{1}{x+y-2} < 1-\frac{1}{x+y} < 1$. Thus, by fixing $x = d(u)$ and $y = d(v)$, for any $uv \in E(G)$, we have $1-\frac{1}{d(u)+d(v)-2} \leq 1-\frac{1}{d(u)+d(v)}$, which concludes that the upper bound in (11) is always finer than (10) and also it is easy to see that, the upper bound (10) is always finer than (9).

In 2012, Ilić and Zhou [13] proposed a new upper bound for $F(G)$.

Theorem 3.7. (See [13]) *Let G be a graph on n vertices and m edges. Then*

$$F(G) \leq (\Delta + \delta)M_1(G) - 2m\Delta\delta \tag{12}$$

with equality holds if and only if G is regular or bidegreed graph.

By comparing the bounds in (9) and (12) and using the fact

$$\chi^2(G) = F(G) + 2M_2(G)$$

immediately leads to the following corollaries.

Corollary 3.5. *With the assumptions in Theorem 3.7 one has the inequality*

$$\chi^2(G) \leq 2M_2(G) + (\Delta + \delta)M_1(G) - 2m\Delta\delta. \tag{13}$$

Corollary 3.6. *Let G be any simple graph with n vertices and m edges. Then*

$$\chi^2(G) \leq 2F(G) \tag{14}$$

with equality holds if and only if G is regular.

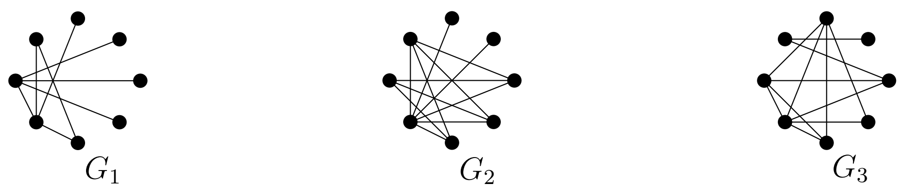

Figure 1. Graphs on 8 vertices.

Remark 3.2. Using the graphs in figure 1, we arrive the conclusion that the above mentioned upper bounds are incomparable as described in the following table:

	$\chi^2(G)$	(8)	(11)	(13)	(14)
G_1	252	304	292	260	296
G_2	696	756	918	792	780
G_3	620	688	732	680	648

4. Identities on $\chi^2(\overline{G})$ and $\overline{\chi}^2(G)$

In 2015, Veylaki, Nikmehr and Tavallaee [20] presented the identities for $\chi^2(\overline{G})$ and $\overline{\chi}^2(G)$.

Proposition 4.1. [20] *Let G be a simple graph with n vertices and m edges. Then*

$$\chi^2(\overline{G}) = 4(n-1)^2\overline{m} - 4(n-1)\overline{M}_1^2(G) + \overline{\chi}^2(G) \tag{15}$$

$$\overline{\chi}^2(G) = 4(n-1)^2\overline{m} - 4(n-1)M_1(\overline{G}) + \chi^2(\overline{G}) \tag{16}$$

The above mentioned identities gives the relation between $\chi^2(\overline{G})$ and $\overline{\chi}^2(G)$. Still, it is not easy to compute $\chi^2(\overline{G})$ and $\overline{\chi}^2(G)$ for the given simple graph to reach the above identities. Now we present the identities for $\chi^2(\overline{G}), \overline{\chi}^2(G)$ only in-terms of the graph invariants of G.

Proposition 4.2. *Let G be a simple graph with n vertices and m edges. Then*

$$\chi^2(\overline{G}) = 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n-6)M_1(G) - 2M_2(G) - F(G),$$
$$\overline{\chi}^2(G) = 4m^2 + (n-2)M_1(G) - 2M_2(G) - F(G).$$

Proof. By the definition of the general sum-connectivity index with $\alpha = 2$ and using (1), we have

$$\chi^2(\overline{G}) = F(\overline{G}) + 2M_2(\overline{G}).$$

It is easy to see that

$$\begin{aligned} F\left(\overline{G}\right) &= \sum_{uv \in E(\overline{G})} \left[d(u)^2 + d(v)^2\right] = \sum_{v \in V(\overline{G})} d(v)^3 = \sum_{v \in V(G)} (n-1-d(v))^3 \\ &= (n-1)^3 \sum_{v \in V(G)} 1 - 3(n-1)^2 \sum_{v \in V(G)} d(v) + 3(n-1) \sum_{v \in V(G)} d(v)^2 - \sum_{v \in V(G)} d(v)^3 \\ &= n(n-1)^3 - 6m(n-1)^2 + 3(n-1)M_1\left(G\right) - F\left(G\right). \end{aligned}$$

In analogous manner, we have

$$M_1(\overline{G}) = n(n-1)^2 - 4m(n-1) + M_1(G).$$

Mansour and Song have given the generalized version for $M_1^{\alpha}(\overline{G})$ in [16] and from [7], we have the following identity

$$M_2\left(\overline{G}\right) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \left(n - \frac{3}{2}\right)M_1(G) - M_2\left(G\right).$$

Using the above identity along with $F(\overline{G})$ in $\chi^2(\overline{G})$, we get

$$\chi^2(\overline{G}) = 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 + (5n-6)M_1(G) - 2M_2(G) - F(G).$$

Finally, substituting $\chi^2(\overline{G})$ and $M_1(\overline{G})$ in (16) completes the proof. $\square$

5. Conclusion

In this paper, we determine few upper bounds for the hyper Zagreb index using forgotten topological index. Along in this line, determining new lower bounds for hyper Zagreb index using second Zagreb index and forgotten index are considered to be studied in future.

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References

[1] A.R. Ashrafi, T. Došlić and A. Hamzeha, The Zagreb coindices of graph operations, *Discrete Appl. Math.* **158** (2010), 571–578.

[2] M. Bianchi, A. Cornaro, J.L. Palacios and A. Torriero, New bounds of degree-based topological indices for some classes of c-cyclic graphs. *Discrete App. Math.* **184** (2015), 62–75.

[3] G.B.A. Xavier, E. Suresh and I. Gutman, Counting relations for general Zagreb indices, *Kragujevac J. Math.* **38** (2014), 95–103.

[4] S.M. Cioabă, Sums of powers of the degrees of a graph, *Discrete Math.* **41** (2006), 1959–964.

[5] D. de Caen, An upper bound on the sum of squares of degrees in a graph, *Discrete Math.* **185** (1998), 245–248.

[6] K.C. Das, Maximizing the sum of the squares of the degrees of a graph. *Discrete Math.* **285** (2004), 57–66.

[7] K.C. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun Math Comput Chem.* **52** (2004), 103–112.

[8] K.C. Das, K. Xu and J. Nam, Zagreb indices of graphs, *Front. Math. China* **10** (2015), 567–582.

[9] J. Devillers and A.T. Balaban, *Topological Indices and Related Descriptors in QSAR and QSPR, Gordon and Breach, Amsterdam*, (1999).

[10] S. Fajtlowicz, On conjectures of graffiti II, *Congr. Numer.* **60** (1987), 189–197.

[11] B. Furtula and I. Gutman, A forgotten topological index, *J. Math. Chem.* **53** (2015), 1184–190.

[12] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972), 535–538.

[13] A. Ilić and B. Zhou, On reformulated Zagreb indices, *Discrete Appl. Math.* **160** (2012), 204–209.

[14] X. Li and J. Zheng, A unified approach to the extremal trees for different indices, *MATCH Commun. Math. Comput. Chem.* **54** (2005), 195–208.

[15] M. Karelson, *Molecular Descriptors in QSAR/QSPR, Wiley-Interscience, New York,* (2000).

[16] T. Mansour and C. Song, The a and $(a, b)-$ Analogs of Zagreb Indices and Coindices of Graphs, *Intern. J. Combin.* (2012), ID 909285.

[17] T. Mansour, M.A. Rostami, E. Suresh and G.B.A. Xavier, New sharp lower bounds for the first Zagreb index. *Scientific Publications of the State University of Novi Pazar Series A* **8** (2016), 11–19.

[18] H. Narumi, New topological indices for the finite and infinite systems.*MATCH Commun. Math. Comput. Chem.* **22** (1987), 195–207.

[19] G.H. Shirdel, H. Rezapour and A.M. Sayadi, The Hyper-Zagreb Index of Graph Operations, *Iranian J. Math. Chem.* **4** (2013), 213–220.

[20] M. Veylaki, M.V. Nikmehr and H.A. Tavallaee, The third and Hyper-Zagreb coindices of some graph operations, *J. Appl. Math. Compt.* **50** (2016), 315–325.

[21] K. Xu and K.C. Das, Some extremal graphs with respect to inverse degree, *Discrete Appl. Math.* **203:20** (2016), 171–183.

[22] B. Zhou and N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.* **47** (2010), 210–218.

[23] B. Zhou and N. Trinajstić, Some properties of the reformulated Zagreb indices, *J. Math. Chem.* **48** (2010), 714–719.