

Considerations on the Reynolds' Transport Theorem

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Abstract— The Reynolds' transport theorem deals with the rate of change of an extensive property, N , of a fluid in a control volume. Its purpose is to provide a link between the concepts associated to the control volumes and those associated to systems. The Reynolds' transport theorem is something extremely important in the formulation of the basic laws of fluid dynamics, which are the mass conservation equation, momentum conservation equations and the energy conservation equation. This paper aims to propose an approach of the Reynolds' Transport Theorem for finite control volume equations for mass, momentum and energy.

Keywords—Control Volume, System, Reynolds' Transport.

I. INTRODUCTION

In general, the basic laws that the movement of fluids obey are enunciated and therefore lead to the motion equation. Often in the study of fluid flow it is preferred an approach from the control volume because it is easier and very relevant to study the movement of fluids. The question being asked is "How to connect the basic laws to a system with a control volume approach to fluids?". This issue has already been predicted by many. The result is the so-called Reynolds' transport theorem, which relates derivatives of system properties to the control volume formulation.

The equations for mass, energy and momentum are associated to a system, and we now want to "convert" these equations into equivalent equations for control volume. For this, we will use the symbol N to represent any of the extensive properties of the system. We can imagine N as related to an amount of mass, linear motion, angular motion, or system energy.

The corresponding intensive property (N/m) will be denoted by η . The relationship between the rate of change of an arbitrary extended property, N , of a system and the property variations within a control volume is given by the following equation (1), known as the Reynolds' transport theorem

$$\left. \frac{dN}{dt} \right|_{system} = \frac{\partial}{\partial t} \int_{VC} \eta \rho dV + \int_{SC} \rho \eta \vec{V} \cdot d\vec{A} \quad (1)$$

The physical interpretation of each of the terms can be found in several textbooks of fluid mechanics, some of them cited in the references (1,2,3) and it follows below:

$\left. \frac{dN}{dt} \right|_{system}$ Represents the total rate of change of an arbitrary extensive property of the system

$\frac{\partial}{\partial t} \int_{VC} \eta \rho dV$ represents the rate of time change of the arbitrary extensive property, N , within the control volume η represents the intensive property corresponding to N (per unit mass)

ρdV represents a mass element contained in the control volume

$\int_{VC} \eta \rho dV$ represents the total amount of the extensive property, N , contained within the control volume

$\int_{SC} \eta \rho \vec{V} \cdot d\vec{A}$: represents the total flow of the general property, N , through the control surface

$\rho \vec{V} \cdot d\vec{A}$ represents the mass of fluid flowing through the area element in the time unit

$\eta \rho \vec{V} \cdot d\vec{A}$ represents the rate of outflow of the extensive property, N , through the area

At this point it is better to make dV the volume differential as not to be confused with the velocity V .

II. MASS CONSERVATION

The first physical principle to which we apply the relationship between system formulations and control volume is the mass conservation principle.

The mass of a system remains constant. According to the considerations made in Eq. (1) and, by making $N = M$ and $\eta = 1$, we have:

$$\frac{dm_{system}}{dt} = \frac{\partial}{\partial t} \int_{VC} \rho dV + \int_{SC} \rho(\vec{V} \cdot \vec{n}) dA \quad (2)$$

For $\frac{dm}{dt} = 0$ we have the mass conservation expressed by

$$0 = \frac{\partial}{\partial t} \int_{VC} \rho dV + \int_{SC} \rho \vec{V} \cdot d\vec{A}, (3)$$

as can be seen from references (1), (2) and (3).

III. FLOW IN PERMANENT REGIME

They are flows that do not vary with time, it cannot vary in a certain point, in a certain time, that is, their characteristics and their properties are permanent over time. Therefore

$$\frac{\partial}{\partial t} \int_{VC} \rho dV = 0, (4)$$

with this comes

$$0 = \int_{SC} \rho \vec{V} \cdot d\vec{A}$$

Solving the integral in question we have

$$\int_{SC} \rho \vec{V} \cdot d\vec{A} = \rho V_o A_o - \rho V_i A_i = 0 (5)$$

$$\vec{F} = \left. \frac{d\vec{P}}{dt} \right)_{system}, (7)$$

where \vec{P} represents the linear momentum of the system. The resulting force includes all field and surface forces

$$\vec{F} = \vec{F}_F + \vec{F}_S$$

Considering the Reynolds' transport theorem given by equation (1)

$$\left. \frac{dN}{dt} \right)_{system} = \frac{\partial}{\partial t} \int_{VC} \eta \rho dV + \int_{SC} \rho \eta \vec{V} \cdot d\vec{A},$$

and by making $N = \vec{P}$ and $\eta = \vec{V}$, it follows that

$$\left. \frac{d\vec{P}}{dt} \right)_{system} = \frac{\partial}{\partial t} \int_{VC} \vec{V} \rho dV + \int_{SC} \rho \vec{V} \cdot \vec{V} \cdot d\vec{A} \quad (8)$$

As in the initial instant, the system and the VC coincide, from equations (1), (2) and (4) we have:

$$\left. \frac{d(m\vec{V})}{dt} \right)_{system} = \vec{F} = \vec{F}_F + \vec{F}_S = \frac{\partial}{\partial t} \int_{VC} \vec{V} \rho dV + \int_{SC} \rho \vec{V} \cdot \vec{V} \cdot d\vec{A} (9)$$

For uniform and permanent flow:

It can be concluded that the product of the density by the input area and the input speed is equal to the density times the output area and the output speed. Such a condition leaves the flow in equilibrium; that is, the input flow equals output flow.

IV. INCOMPRESSIBLE FLOW

In some cases, it is possible to simplify the previous equation, as in the case of an incompressible flow (specific mass $\rho = \text{constant}$, generally valid for liquids). When ρ does not depend either on space or time, the equation can be written as:

$$\int_{SC} \vec{V} \cdot d\vec{A} = 0 (6)$$

In uniform flow it implies that the velocity is constant across the entire section area. If, in addition, ρ is also constant in the section, it results

$$V_2 A_2 = V_1 A_1$$

EQUATION OF THE LINEAR MOMENTUM CONSERVATION FOR AN INERTIAL CONTROL VOLUME

This analysis is restricted to an inertial control volume, that is, there is not acceleration relative to a steady reference system or inertial coordinate system. The following text can be verified by references (1), (2) and (3). Recalling Newton's second law for a system:

$$\frac{\partial}{\partial t} \int_{VC} \vec{V} \rho dV = 0$$

This allows us to write that

$$\vec{F} = \rho_2 A_2 V_2 \vec{V}_2 - \rho_1 A_1 V_1 \vec{V}_1$$

On the other hand, taking into account the continuity equation we have

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m},$$

where $\dot{m} = \frac{dm}{dt}$

The formulation for VC of the Newton's second law is given by

$$F = \dot{m} (V_2 - V_1) \quad (10)$$

ENERGY EQUATION FOR AN INERTIAL CONTROL VOLUME

Again, starting from equation (1) and references (1), (2) and (3) we have

$$\left(\frac{dN}{dt} \right)_{system} = \frac{\partial}{\partial t} \int_{VC} \eta \rho dV + \int_{SC} \rho \eta \vec{V} \cdot d\vec{A}$$

By making $\eta = e$ and $N = E$, it comes that

$$\left(\frac{dE}{dt} \right)_{system} = \frac{\partial}{\partial t} \int_{VC} e \rho dV + \int_{SC} \rho e \vec{V} \cdot d\vec{A} \quad (11)$$

On the other hand, it is known that

$$e = e_{internal} + e_{kinetic} + e_{potential}$$

$$e = u + \frac{V^2}{2} + gz$$

$$\left(\frac{dE}{dt} \right)_{system} = \dot{Q} - \dot{W} \quad (12)$$

Therefore

$$\left(\frac{dE}{dt} \right)_{system} = \dot{Q} - \dot{W} = \frac{\partial}{\partial t} \int_{VC} \left(u + \frac{V^2}{2} + gz \right) \rho dV + \int_{SC} \left(u + \frac{V^2}{2} + gz \right) \rho \vec{V} \cdot d\vec{A} \quad (13)$$

We know that, for a permanent flow:

$$\frac{\partial}{\partial t} \int_{VC} \left(u + \frac{V^2}{2} + gz \right) \rho dV = 0,$$

What implies in

$$\dot{Q} - \dot{W} = \int_{SC} \left(u + \frac{V^2}{2} + gz \right) (\rho \vec{V} \cdot \vec{n}) dA \quad (14)$$

Finally, for a non-deformable control volume, it may be written

$$\int_{SC} \left(u + \frac{V^2}{2} + gz \right) (\rho \vec{V} \cdot \vec{n}) dA = \left[\left(u + \frac{V^2}{2} + gz \right) (\rho VA) \right]_2 - \left[\left(u + \frac{V^2}{2} + gz \right) (\rho VA) \right]_1 \quad (15)$$

where

$$(\rho VA) = \dot{m}$$

V. RELATIONSHIP BETWEEN THE ENERGY EQUATION AND THE BERNOULLI'S EQUATION

Starting from the energy equation and the references (1), (2) and (3), it may be written

$$\left(\frac{dE}{dt}\right)_{system} = \dot{Q} - \dot{W} = \frac{\partial}{\partial t} \int_{VC} e \rho dV + \int_{SC} \left(e + \frac{P}{\rho}\right) \rho \vec{V} \cdot d\vec{A} \quad (16)$$

By supposing that $\dot{W} = 0$ and by considering the permanent regime

$$\frac{\partial}{\partial t} \int_{VC} \vec{V} \rho dV = 0,$$

We have that

$$\dot{Q} = \int_{SC} \left(e + \frac{P}{\rho}\right) \rho \vec{V} \cdot d\vec{A} \quad (17)$$

By developing the integral on the SC control surface, we have

$$\dot{Q} = \int_1 \left(e + \frac{P}{\rho}\right) \rho \vec{V} \cdot d\vec{A} + \int_2 \left(e + \frac{P}{\rho}\right) \rho \vec{V} \cdot d\vec{A}$$

More specifically, by solving the integral the equation becomes

$$\dot{Q} = \left(u_1 + gz_1 + \frac{V_1^2}{2} + \frac{p_1}{\rho_1}\right) (-\rho_1 V_1 A_1) + \left(u_2 + gz_2 + \frac{V_2^2}{2} + \frac{p_2}{\rho_2}\right) (\rho_2 V_2 A_2)$$

Taking into account the mass conservation equation

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m},$$

we have

$$\dot{Q} = \left(u_1 + gz_1 + \frac{V_1^2}{2} + \frac{p_1}{\rho_1}\right) (-\dot{m}) + \left(u_2 + gz_2 + \frac{V_2^2}{2} + \frac{p_2}{\rho_2}\right) (+\dot{m})$$

This equation can also be written

$$\left(gz_1 + \frac{V_1^2}{2} + \frac{p_1}{\rho_1}\right) = \left(gz_2 + \frac{V_2^2}{2} + \frac{p_2}{\rho_2}\right) + \left(u_2 - u_1 - \frac{\dot{Q}}{\dot{m}}\right)$$

For reversible adiabatic processes, it implies the nullity of the following term:

$$\left(u_2 - u_1 - \frac{\dot{Q}}{\dot{m}}\right)$$

In view of this, one arrives at Bernoulli's equation

$$\left(gz_1 + \frac{V_1^2}{2} + \frac{p_1}{\rho_1}\right) = \left(gz_2 + \frac{V_2^2}{2} + \frac{p_2}{\rho_2}\right)$$

VI. FINAL CONSIDERATIONS

The derivation of the Reynolds' transport theorem (Rtt) may seem very complex, but when the basis of the theorem is understood, it is indeed easy to follow its derivation. We should start with a system and the rate at which an extended property N changes in it.

In most contemporary textbooks these equations are derived by transforming the corresponding equations into a control mass using the Reynolds' transport theorem. This theorem is mathematically correct, and mastering its

derivation is a good mathematical exercise, but Rtt is a difficult medium for this purpose, since learning and mastering Rtt is not an end in itself, particularly in an introductory course. In addition, it is comforting to have evidence that the laws governing the control masses can be translated into the laws governing the contents of the control volumes.

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