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On Fuzzy Length Spaces and Fuzzy Geodesics

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Abstract*— *In this paper, the notion of a fuzzy length space is introduced, and the definition of a fuzzy geodesic in a fuzzy metric space is given. Some properties of these new concepts are studied and the relation between them is revealed.

Keywords*— *Fuzzy sets, Fuzzy length spaces, Fuzzy geodesics, Fuzzy metric spaces.

I. INTRODUCTİON

The theory of fuzzy sets has attracted many researchers since it's been introduced by L.A. Zadeh [1]. The theory has advanced in many disciplines including artificial intelligence, control engineering, decision theory, computer science, robotics. No wonder, mathematical developments also have advanced to a very high standard with fuzzy sets. The theory has influenced almost every branches of classical mathematics such as algebra, graph theory, topology and so on by generalizing them. These developments is still ongoing. In recent years, studies concentrated on fuzzy metric spaces and fuzzy topology. Some of them could be found in the papers [2-6].

Recently, the authors in [7] define an arc length notion of continuous curves in fuzzy metric spaces based on the addition operator between fuzzy sets. Their study is the first step to fuzzify the classical metric geometry [8] and to solve some optimization problems. This development gives us an idea to introduce the concept of a fuzzy length space and to investigate fuzzy geodesics which are important for the application of fuzzy metric spaces.

In this study, the notion of a fuzzy length space is introduced, based on the arclength of continuous curves in fuzzy metric spaces. We give a definition for fuzzy geodesics in fuzzy metric spaces. Some properties of these new concepts are studied and therelation between fuzzy length spaces and fuzzy geodesic spaces is investigated.

II. PRELİMİNARİES

In this section, we would like to give some basic definitions and properties.

Definition 2.1. [7] A continuous $t-$norm is a binary operator $*: [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:

(T1) $*$ is associative and commutative,
(T2) $*$ is continuous,
(T3) $a * 1 = a$ for all $a \in [0,1]$,
(T4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, where $a, b, c, d \in [0,1]$.

Example 2.2.

(i) If $a * b = a \times b$, where $\times$ is the scalar multiplication operator, then $*$ is a continuous $t-$norm.
(ii) If $a * b = min\{a, b\}$, where min is the minimum operator, then $*$ is a continuous $t-$norm.

Definition 2.3. [9] The $3-$tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times \mathbb{R}^+$ satisfying the following conditions:

(M1) $M(x, y, t) > 0$,
(M2) $M(x, y, t) = 1$ iff $x = y$,
(M3) $M(x, y, t) = M(y, x, t)$,
(M4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(M5) $M(x, y, .): \mathbb{R}^+ \to [0,1]$ is continuous,

$x, y, z \in X$ and $t, s \in \mathbb{R}^+$. M is called a fuzzy metric on X. The value $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.4. [9] Let $(X, M, *)$ be a fuzzy metric space. An open ball $B(x, r, t)$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$ is defined as

$$B(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Then, a topology on X can be defined as

$$\tau = \{A \subset X | x \in A \; iff \; there \; exists \; t \in \mathbb{R}^+ \; and \; r, 0 < r < 1 \; such \; that \; B(x, r, t) \subset A\}.$$

A map $\alpha: [a, b] \to X$ in this space is continuous if and only if for any $A \in \tau$ there exists an open interval $I \subset \mathbb{R}$ such that $\alpha^{-1}(A) = I \cap [a, b]$. A continuous curve in $(X, M, *)$ is a continuous map $\alpha: [a, b] \to X$.

Definition 2.5. [7] Let $(X, M, *)$ be a fuzzy metric space, and $\alpha: [a, b] \to X$ be a continuous curve in $(X, M, *)$. Define the map $\mathcal{L}_M(\alpha|_{[a,b]}): \mathbb{R}^+ \to [0,1]$ by

$$\mathcal{L}_M(\alpha|_{[a,b]})(t) = \inf_{m \in \mathbb{N}, \{t_i\} \in \mathcal{P}^m([a,b])} \left(\bigoplus_{i=1}^{m} \widetilde{M}(\alpha(t_{i-1}), \alpha(t_i)) \right)(\lambda) \quad (1)$$

for each $\lambda \in \mathbb{R}^+$, where $\widetilde{M}(x,y)(.)$ is the map $M(x,y,.)$ and $\{t_i\} \in \mathcal{P}^m([a,b])$ is a partition of the interval $[a,b]$. The map $\mathcal{L}_M(\alpha|_{[a,b]})$ is called the arclength of curve α in (X,M,ast). If $\mathcal{L}_M(\alpha|_{[a,b]}) > \mathbf{0}$, where $\mathbf{0}$ is the constant function $A: \mathbb{R}^+ \to [0,1]$ of value 0, then α is called a rectifiable curve in $(X,M,*)$.

Theorem 2.6. [7] Let $(X,M,*)$ be a fuzzy metric space, and $\alpha: [a,b] \to X$ be a continuous curve in $(X,M,*)$. Then, $\mathcal{L}_M(\alpha|_{[a,c]}) \oplus \mathcal{L}_M(\alpha|_{[c,b]}) = \mathcal{L}_M(\alpha|_{[a,b]})$ for any $c \in [a,b]$.

For further reading about the arclength properties of continuous curves in fuzzy metric spaces, please see [7].

III. FUZZY LENGTH SPACES

Definition 3.1. Let $(X,M,*)$ be a fuzzy metric space. If for every $x,y \in X$ there exists a rectifiable curve α in $(X,M,*)$, then $(X,M,*)$ is called a rectifiable fuzzy metric space.

Definition 3.2. Let $(X,M,*)$ be a rectifiable fuzzy metric space. The intrinsic fuzzy metric induced by M is the function $\mu: X \times X \times \mathbb{R}^+ \to [0,1]$ defined by

$$\mu(x,y,\lambda) = sup\{\mathcal{L}_M(\alpha|_{[a,b]}) | \alpha \text{ is a rectifiable curve in } X, \alpha(a) = x, \alpha(b) = y\},$$

where the supremum is taken over all rectifiable curves from x to y.

Remark 3.3. The condition of being a rectifiable fuzzy metric space in Definition 3.2 is given for making μ satisfy $(M1)$.

Definition 3.4. A fuzzy length space is a rectifiable fuzzy metric space whose metric is intrinsic.

Proposition 3.5. Let $(X,M,*)$ be a rectifiable fuzzy metric space. Then we have $\tilde{\mu}(x,y) \leq \widetilde{M}(x,y)$.

Proof. We know that for every rectifiable curve α in $(X,M,*)$, we have the inequality $\mathcal{L}_M(\alpha|_{[a,b]}) \leq \widetilde{M}(x,y)$ [?]. Then, $\widetilde{M}(x,y)(\lambda)$ is an upper bound for the set

$$\{\mathcal{L}_M(\alpha|_{[a,b]}) | \alpha \text{ is a rectifiable curve in } X, \alpha(a) = x, \alpha(b) = y\}.$$

Hence we get

$$\tilde{\mu}(x,y)(\lambda) = \mu(x,y,\lambda) = sup\{\mathcal{L}_M(\alpha|_{[a,b]}) | \alpha \text{ is a rectifiable curve in } X, \alpha(a) = x, \alpha(b) = y\} \leq \widetilde{M}(x,y)(\lambda).$$

Proposition 3.6. Let $(X,M,*)$ be rectifiable fuzzy metric space, and μ be the intrinsic fuzzy metric induced by M. If α is a rectifiable curve in $(X,M,*)$, then $\mathcal{L}_\mu(\alpha|_{[a,b]}) = \mathcal{L}_M(\alpha|_{[a,b]})$.

Proof. By definition of the arclength

$$\begin{aligned}\mathcal{L}_\mu(\alpha|_{[a,b]}) &= \inf_{m\in\mathbb{N},\{t_i\}\in\mathcal{P}^m([a,b])} \left(\oplus_{i=1}^m \tilde{\mu}(\alpha(t_{i-1}),\alpha(t_i))\right)(\lambda) \\ &\leq \inf_{m\in\mathbb{N},\{t_i\}\in\mathcal{P}^m([a,b])} \left(\oplus_{i=1}^m \widetilde{M}(\alpha(t_{i-1}),\alpha(t_i))\right)(\lambda) \\ &= \mathcal{L}_M(\alpha|_{[a,b]}).\end{aligned}$$

Thus we get $\mathcal{L}_\mu(\alpha|_{[a,b]}) \leq \mathcal{L}_M(\alpha|_{[a,b]})$. To prove the inverse inequality, let $\{t_i\} \in \mathcal{P}^m([a,b])$ be an arbitrary partition of $[a,b]$. $\mu(\alpha(t_{i-1}),\alpha(t_i),\lambda) \geq \mathcal{L}_M(\alpha|_{[t_{i-1},t_i]})$, since the left hand side is the supremum of lengths one of which is written on the right hand side. Therefore,

$$\begin{aligned}\mathcal{L}_\mu(\alpha|_{[a,b]})(\lambda) &= \inf_{m\in\mathbb{N},\{t_i\}\in\mathcal{P}^m([a,b])} \left(\oplus_{i=1}^m \tilde{\mu}(\alpha(t_{i-1}),\alpha(t_i))\right)(\lambda) \\ &\geq \mathcal{L}_M(\alpha|_{[a,b]})(\lambda).\end{aligned}$$

Thus we get $\mathcal{L}_\mu(\alpha|_{[a,b]}) \geq \mathcal{L}_M(\alpha|_{[a,b]})$.

□

IV. FUZZY GEODESİCS

Definition 4.1. Let $(X,M,*)$ be a fuzzy metric space, and $\alpha: [a,b] \to X$ be a continuous curve in $(X,M,*)$. α is said to have fuzzy constant speed if there exist a constant $k > 0$ and a fuzzy set κ on $\mathbb{R}^+$, such that $\mathcal{L}_M(\alpha|_{[t,t']})(k) = \kappa(|t-t'|)$ for all $t,t' \in [a,b]$, $t > t'$. The fuzzy set κ is called the fuzzy speed of α. α is said to have fuzzy unit speed κ if $k = 1$.

Example 4.2. Let (X,d) be a metric space, and $\alpha: [a,b] \to X$ be a continuous curve in (X,d). We know that the curve α is a continuous curve in the fuzzy metric space (X,M,min), where $M(x,y,\lambda) = \frac{\lambda}{\lambda + d(x,y)}$ with the arclength $\mathcal{L}_M(\alpha|_{[a,b]})(\lambda) = \frac{\lambda}{\lambda + \mathcal{L}_d(\alpha|_{[a,b]})}$ in (X,M,min), where $\mathcal{L}_d(\alpha|_{[a,b]})$ is the arclength of the curve α in (X,d) [7]. Let α has constant speed in (X,d). Then, there exists a constant $m > 0$ such that $\mathcal{L}_d(\alpha|_{[t,t']}) = m|t-t'|$ for all $t,t' \in [a,b]$, $t > t'$. For the fuzzy set $\kappa: \mathbb{R}^+ \to [0,1]$, $\kappa(x) = \frac{1}{1+x}$, we have $\mathcal{L}_M(\alpha|_{[t,t']})(k) = \kappa(|t-t'|)$, where $k = m > 0$. Therefore α has fuzzy constant speed κ in (X,M,min). If α has unit speed in (X,d), then it has fuzzy unit speed in (X,M,min).

Example 4.3. Let (X,d) be a metric space, and $\alpha:[a,b]\to X$ be a continuous curve in (X,d). We know that the curve α is a continuous curve in the fuzzy metric space $(X,M,\times)$, where $M(x,y,\lambda)=e^{\frac{-d^2(x,y)}{\lambda}}$ with the arclength $\mathcal{L}_M(\alpha|_{[a,b]})(\lambda)=e^{\frac{-\mathcal{L}_d^2(\alpha|_{[a,b]})}{\lambda}}$ in $(X,M,\times)$, where $\mathcal{L}_d(\alpha|_{[a,b]})$ is the arclength of the curve α in (X,d) [7]. Let α has constant speed in (X,d). Then α has fuzzy constant speed in $(X,M,\times)$ with the fuzzy speed $\kappa(x)=e^{-x^2}$.

Example 4.4. Let (X,d) be a metric space, and $\alpha:[a,b]\to X$ be a continuous curve in (X,d). We know that the curve α is a continuous curve in the fuzzy metric space $(X,M,\times)$, where $M(x,y,\lambda)=e^{-d(x,y)}$ with the arclength $\mathcal{L}_M(\alpha|_{[a,b]})(\lambda)=e^{-\mathcal{L}_d(\alpha|_{[a,b]})}$ in $(X,M,\times)$, where $\mathcal{L}_d(\alpha|_{[a,b]})$ is the arclength of the curve α in (X,d) [7]. Then, α does not have the fuzzy constant speed in $(X,M,\times)$.

Definition 4.5. Let $(X,M,*)$ be a fuzzy metric space. A continuous curve $\alpha:[a,b]\to X$ is called a fuzzy geodesic in $(X,M,*)$ if α has fuzzy constant speed and if $\mathcal{L}_M(\alpha|_{[t,t']})=\widetilde{M}(\alpha(t),\alpha(t'))$ for all $t,t'\in[a,b]$, $t>t'$.

Example 4.6. Let (X,d) be a metric space, and $\alpha:[a,b]\to X$ be a geodesic in (X,d). Then α is a fuzzy geodesic in the fuzzy metric space (X,M,min) where $M(x,y,\lambda)=\frac{\lambda}{\lambda+d(x,y)}$.

Definition 4.7. A fuzzy metric space $(X,M,*)$ is called a fuzzy geodesic space if for every pair of points $x,y\in X$ there exists a geodesic $\alpha:[a,b]\to X$ joining x to y.

Proposition 4.8. Let $(X,M,*)$ be a fuzzy metric space, and $\alpha:[a,b]\to X$ be a continuous curve in $(X,M,*)$. α is a fuzzy geodesic if and only if there exist a constant $k>0$ and a fuzzy set κ, such that $M(\alpha(t),\alpha(t'),k)=\kappa(|t-t'|)$ for all $t,t'\in[a,b]$, $t>t'$.

Proof. The proof is straight forward.

□

Theorem 4.9. Let $(X,M,*)$ be a rectifiable fuzzy metric space. If $(X,M,*)$ is a fuzzy geodesic space, then $(X,M,*)$ is a fuzzy length space.

Proof. By Proposition 3.5 $\tilde{\mu}(x,y)\le\widetilde{M}(x,y)$. Thus, it's sufficient to show that $\widetilde{M}(x,y)\le\widetilde{M}(x,y)$. Now for every $x,y\in X$, there exists a fuzzy geodesic α from x to y. Then, for any $\lambda\in\mathbb{R}^+$

$$\begin{aligned}
M(x,y,\lambda)=\widetilde{M}(x,y)(\lambda)&=\widetilde{M}(\alpha(a),\alpha(b))(\lambda)\\
&=\mathcal{L}_M(\alpha|_{[a,b]})(\lambda)=\mathcal{L}_\mu(\alpha|_{[a,b]})(\lambda)\\
=\inf_{m\in\mathbb{N},\{t_i\}\in\mathcal{P}^m([a,b])}&\left(\bigoplus_{i=1}^{m}\tilde{\mu}(\alpha(t_{i-1}),\alpha(t_i))\right)(\lambda)\\
&\le\tilde{\mu}(\alpha(a),\alpha(b))(\lambda)\\
&=\mu(\alpha(a),\alpha(b),\lambda)=\mu(x,y,\lambda).
\end{aligned}$$

Thus we get $\widetilde{M}(x,y)\le\tilde{\mu}(x,y)$.

□

V. CONCLUSİON

In our study, the notion of a fuzzy length space is introduced, based on the arclength of continuous curves in fuzzy metric spaces. We give a definition for fuzzy geodesics in fuzzy metric spaces. For the relation between fuzzy geodesic spaces and fuzzy length spaces, a theorem is given and our theorem shows that the relation is similar with the corresponding ones in metric spaces, but not the same. Our findings would be useful for the possible applications of fuzzy geodesic and fuzzy length spaces, including optimization problems.

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