

# Five New Ways to Prove a Pythagorean Theorem

Nurul Laily, Hobri, Dafik

Department of Mathematics Education, University of Jember, Indonesia

**Abstract**— Pythagoras is one of the mathematicians who developed the basic theories of mathematics. One of his taunts that are well-known even by primary school students is a Pythagorean Theorem. This theorem states that in a right-angled triangle, the square of the hypotenuse is equal to the sum of each other sides square. There are many proofs which have been developed by a scientist, we have estimated up to 370 proofs of the Pythagorean Theorem. In this paper, we are trying to develop five new proofs of Pythagorean Theorem by using algebraic-geometric proof. The first proof is proven by the trapezoidal shape constructed by five right triangles. The second and third Proofs are proven by using the constructed parallelograms consisting four right triangles and two isosceles trapezoids. The fourth proof is proven by trapezoidal shape constructed of three pieces of a congruent trapezoid, and the fifth proof is proven by using a rectangle constructed by congruent square. Thus, we conclude that the proof of the Pythagorean Theorem can be proven by using the construction of flat trapezoid, parallelogram, square, and rectangular by means of a right-angle triangle.

**Keywords**— Pythagoras theorem, right-angle triangle, Trapezoid, Square, Rectangle..

## I. INTRODUCTION

Pythagorean Theorem becomes an important base in the calculation of the length side of the flat straight sides with the help of right-angled triangles, because the Pythagorean Theorem is a fundamental theorem in mathematics. The Pythagorean Theorem has been introduced to students from elementary school until secondary school. Pythagoras discovery in the field of music and mathematics remains alive today. Pythagorean Theorem is taught in schools and used to calculate the distance a side of a right triangle. Before Pythagoras, there were no proofs or assumptions underlying on right triangle systematic. Pythagoras was the first person who coined that axioms, postulates outlined in advance in developing geometry.

The famous Pythagorean Theorem states that the square of the hypotenuse of a right triangle is equal to the sum of each other sides square. Although the development of the various version of other proofs has been widely known

before Pythagoras, all proof related to the right triangle is still addressed to the Pythagorean Theorem, because he was the first to prove the observation mathematically in a right triangle.

One of the benefits of this theorem is as a tool in the calculation of the natural phenomena. The Pythagorean Theorem was a base of proving Fermat's theorem in 1620:  $xn + yn = zn$ , which was firstly proven by Sir Andrew Wiles in 1994. After that, some math calculations in a quite complicated technique were resolved.

The book: *The Pythagorean Proposition* written by ES Loomis, second edition published since 1940 is a major collection of proofs of Pythagoras theorem. This book has been collected as many as 370 different proofs of Pythagorean Theorem. A proof given by Euclid, as well as a modern mathematician like Legendre, Leibniz and Huygens, also ex-president of the United States (James Garfield) have enriched the collection of that book.

In that book, Loomis classify all evidence into four categories. Most evidence are categorized in algebraic or geometric proof. In algebra, the proof of this theorem is shown by the number of squares of the two legs lengths which are equal to the length of the hypotenuse. While the geometrical proof is indicated by the box area developed from two foot square areas that are equal, to that, produced on the hypotenuse. That book consists of 109 algebraic proof and 255 geometric proof geometric. (There are also 4 "quaternionic" evidences and two "dynamic" evidences). Furthermore, the total calculation of all category, the proof is reaching 370. In addition there are five journal latest evidences, such as the evidence in the category of geometry-trigonometry. Therefor, the total is 375 evidentiary proof.

The spread of Pythagoras Theorem is very fast. So many books and internet portals review this theorem and its proof. Since the days of Pythagoras, many different proofs of the Pythagorean Theorem are published. In the second printed book: *"The Pythagorean Proposition"* Lomis ES 370 has collected and classified the evidence of this famous theorem.

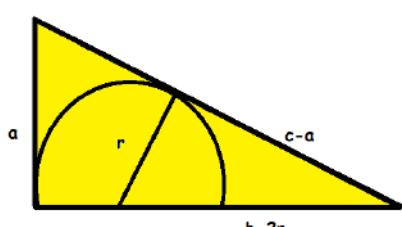
Proofs of Pythagoras that have been found by researchers are a five-proof algebraic - geometric proof. The first proof is given in a trapezoidal ABGD constructed of five pieces of a right triangle, there are two pairs of right-

angled triangles congruent with the side of the base  $a$  and  $(ba)$  to the height of each  $b$  and  $(a+b)$ . Then, the triangle elbow with a base and height  $c$ , is a way to apply  $a^2 + b^2 = c^2$ . The second Proof is known in parallelogram ABCD. It is constructed of four right-angled triangle, two of which are congruent triangle with each pedestal is  $a, c, (ba)$  and the height  $b, c, \text{ and } (b+a)$ . This is a way to apply  $a^2 + b^2 = c^2$ . The third Proof is in parallelogram AELH which is constructed of two pieces of congruent isosceles trapezoid, each trapezoid constructed by six right-angled triangle, two pairs of which are congruent right-angled triangles with base  $a$  and height  $b$ , the two others are a right triangle with the base length and height is  $c$ . It is a way to apply  $a^2 + b^2 = c^2$ . The fourth proof is given in trapezoidal of ABIH square which is constructed from ABCD trapezoid and DCIH square. ABCD is made of EHGFsquare with sides  $c$  and four congruent right-angled triangles with the pedestal  $a$  and high  $b$ . Trapezoid DCIH is formed from three pieces of right-angled triangles, two of which is a right triangle with base  $a$  and height  $b$ , and a right triangle with base and height  $c$ . In proving the fifth, rectangle ABDE is constructed of two congruent square, each square constructed of square PQRS with sides  $(ba)$  and four right-angled triangle with the base  $a$  and  $b$ . It applies that  $a^2 + b^2 = c^2$ . The next Pythagorean Theorem comes from the 20<sup>th</sup> president of US, J.A Garfield in 1876. The area of the trapezoid below can be calculated in two ways, so that the Pythagorean theorem can be proven by trapezoid coincides extension with the three right-angled triangle, then  $a^2 + b^2 = c^2$ .

On the Square ABCD, in which it creates four right-angled triangle with sides  $a, b$ , and  $c$  as the hypotenuse and unknown  $a, b$ , as the triangular straightener.

Is it proven that square ABCD equals the number of extensive third-angled triangle and square PQRS, then apply  $a^2 + b^2 = c^2$ .

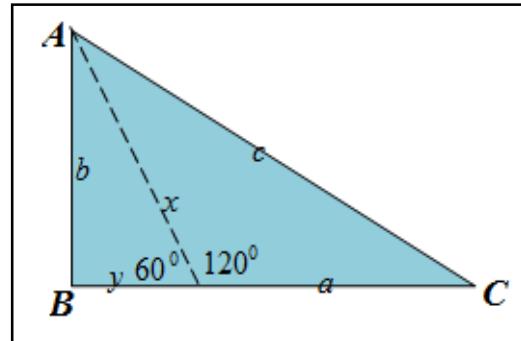
The next theorem was proven by *J. Molokach*, on May 19, 2015 by writing a half-circle to the right-angled triangle.  $r$  is the radius of the semicircle. This is determined by the proportion of (the equality of two triangles).



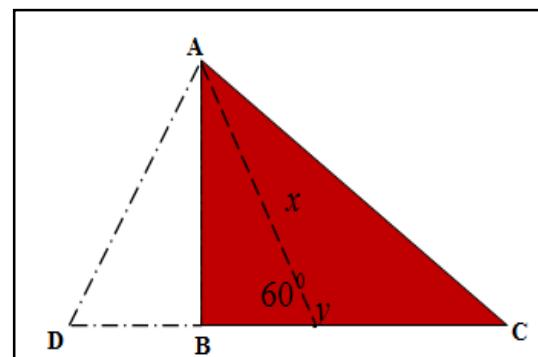
*Proof of J. Molokach*

To prove the Pythagorean Theorem, it is applied to the intersection of two chords in a circle equation, then apply  $a^2 + b^2 = c^2$ .

The Evidence of Pythagoras made by Burkard Polster and Marty Ross published in Mathematics Magazine (VOL. 89, No.1, February 2016 pages:47-54). The Pythagoras proof discovery is an evidence of the cosines law with angles of  $60^\circ$  and  $120^\circ$ . The evidence depends on the Pythagorean Theorem and the general law of cosines.



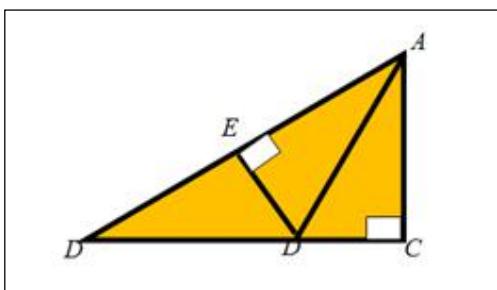
*Pythagoras Burkard Polster- Matty Ross*



*Pythagoras Burkard Polster- Matty Ross*

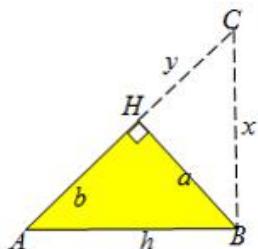
In this case, the evidence refers to the van floor Lamoen discovery derived from the general statement of the Pythagorean Theorem of right isosceles triangle case. Pythagorean Theorem through an angle of  $60^\circ$  and  $120^\circ$ , any right-angled triangle can be divided into one angle of  $60^\circ$  and the other  $120^\circ$  angle. Using the notation in the diagram refers to the results of the Polster and Ross. Pythagorean Theorem through  $60^\circ$  and  $120^\circ$  obtained:  $b^2 + a^2 = c^2$ .

Proofs of Pythagoras have been published in *the American Mathematical Monthly Magazine*, with an editor record (vol.116 2009, October 2009, p 687) Although this evidence does not appear and known, it is the rediscovery of the evidence, which first appeared printed, and the evidence has been presented by Sang Woo Ryoo, a student of Carlisle High School, Carlisle, PA. Loomis took credit for evidence. Figure AD, angle bisector of the angle A, and DE is perpendicular to AB.



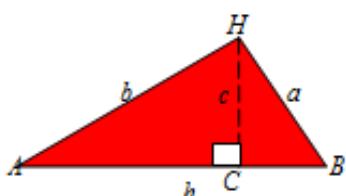
Evidence by Ryoo Sang Woo

the proof leads to  $(cb)(c+b)=A^2$  and to the Pythagoras identity. Thus it obtains the equation:  $b^2+c^2=A^2 \square$ .



Loomis evidence to 16

Given HB is perpendicular to the air conditioner to form three right-angled triangles namely:  $\Delta ABC$ ,  $\Delta AHB$ , and  $\Delta BCH$ . So it uses:  $b^2 + a^2 = h^2$ .



HC is perpendicular to AB. So, to form three right-angled triangle need:  $\Delta ACH$ ,  $\Delta BCH$ ,  $\Delta AHB$ , using the ratio of  $AC: AH: AH: AB$ . then it uses;  $h^2 = a^2 + b^2$ .

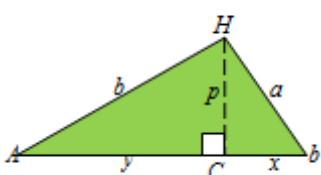
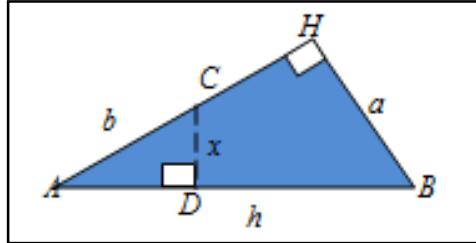


Fig.1: Trapezoid ABDG

By using the ratio:

$$x^2 + p^2 = x^2 + xy = x(x+y) = a^2,$$

it applies,  $h^2 = a^2 + b^2$



An ABC triangle with the elbows in H uses the comparison:

$$\frac{a}{x} = \frac{b}{h-a}; \frac{b}{h-a} = \frac{h}{b-x};$$

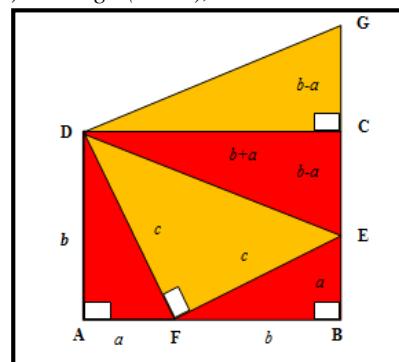
then is obtained:  $a^2 + b^2 = h^2$ .

There are still many ways that can be served to motivate new verification techniques to prove this Pythagorean Theorem.

## II. RESULTS AND DISCUSSION

The following will be presented new evidence in the Pythagoras theorem. Proof of the Pythagorean Theorem developed in this article is categorized in the form of evidence algebraic geometry, wherein each is accompanied by a proof of the theorem, evidence and geometry images to get easier in its presentation. There are four new theorems that found new evidences related to the Pythagoras theorem.

**Theorem 1:** an ABCD trapezoid with a ABDG rectangular and DCG triangle, the ABCD square constructed by  $\triangle DAF$  with right-angled at A with a pedestal, height b, and hypotenuse c. Pulled straight line segment from point F to point B with side lengths b, draw a line segment perpendicular to AB, from point B to point E with a side length. Connect the dots from point E to F with side length c, and from point D to point E, so  $\triangle DFE$  is a right triangle. Pull straight line segment from point E to point C with a long ba. Connect the line segment from point D to point C, draw a line segment from point C to point G with the long side (ba), draw a line segment from point D to point G, so that ABGC is a trapezoid with a base (3b-a) and high (b+a), then shows  $a^2 + b^2 = c^2$ .



**Proof:**

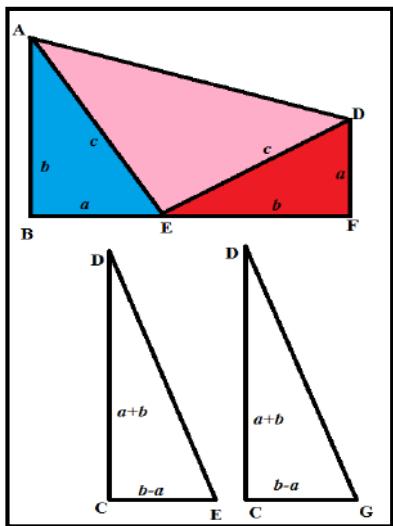


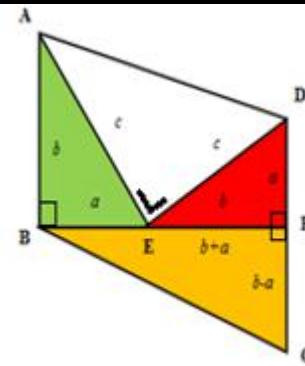
Fig.2: Parallelogram ABCD

The rectangle ABCD is composed of four right-angled triangle that is  $\Delta DAF \cong \Delta FBE$  length and height b,  $\Delta DCE$  have a high pedestal and c, and  $\Delta AFD \cong \Delta FBE$  have a side length and height b. Line segment drawn from point C to point G sided (b-a) and a line segment drawn from point G to point D. As such DCG triangle is a right triangle in C with the length of the base (ba) and high (b+a), then apply  $a^2 + b^2 = c^2$ .

Trapezoid broad ABGD are:

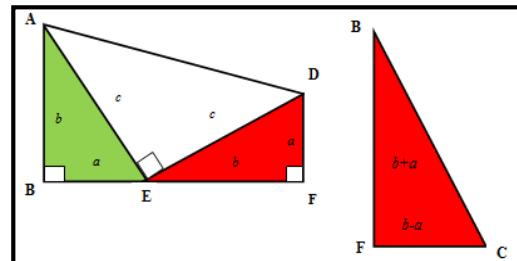
$$\begin{aligned}
 &= \text{Area } \Delta_1 + \text{Area } \Delta_2 + \text{Area } \Delta_3 + \text{Area } \Delta_4 + \text{Area } \Delta_5 \\
 &= \frac{1}{2} \cdot (3b-a) \cdot (b+a) = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} + \frac{b^2-a^2}{2} + \frac{b^2-a^2}{2} \\
 &= \frac{3b^2-ab+3ab-a^2}{2} = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2} + \frac{b^2-a^2}{2} + \frac{b^2-a^2}{2} \\
 &= \frac{3b^2+2ab-a^2}{2} = 2ab+c^2+2b^2-2a^2 \\
 &= b^2+a^2 = c^2
 \end{aligned}$$

**Theorem 2:** a triangle ABE is a right triangle in B, drawn segment of the straight line from point E to point F with the long side b, drawn the line segment perpendicular to BF, from point F to point D with length a, connect line segment from point A to point D and from point D to point E with a length c, draw a line segment perpendicular to BF with point F to point C of length (ba), connect the line segment from point B to point C, such that ABCD is a parallelogram with base (b+a) and b high, then apply  $a^2 + b^2 = c^2$ .



**Proof:**

a parallelogram ABCD consists of trapezoidal ABFD and right-angled triangle BFC. The length of the side AB = b, the DF = a, and high BF = (a + b). Trapezoid ABFD has two pairs of right-angled triangles with the  $\Delta ABE$  and  $\Delta ADF$  high pedestal a and b, and a right triangle with sides of length  $\Delta AED$  high pedestal and is c. Two line segments drawn from point F to point C with side lengths (b-a) and from point B to point C on the side BF.  $\Delta CFB$  is such that the right-angled triangle in F with the high pedestal of a and b, then apply  $a^2 + b^2 = c^2$ .



The vast of parallelogram ABCD =

$$\begin{aligned}
 &\text{Area } \Delta_1 + \text{Area } \Delta_2 + \text{Area } \Delta_3 + \text{Area } \Delta_4 \\
 &= \frac{1}{2}(a+b)(a+b) + \frac{1}{2}(a+b)(b-a) = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab + \frac{1}{2}(b+a)(b-a) \\
 &= \frac{a^2+2ab+b^2}{2} + \frac{b^2-a^2}{2} = \frac{ab}{2} + \frac{c^2}{2} + \frac{ab}{2} + \frac{b^2-a^2}{2} \\
 &= \frac{a^2+2ab+2b^2-a^2}{2} = \frac{c^2+2ab+b^2-a^2}{2} \\
 &= a^2+2ab+2b^2-a^2 = c^2+2ab+b^2-a^2 \\
 &= a^2+b^2 = c^2
 \end{aligned}$$

**Theorem 3:** right triangles ABF and FBC with the base a, height b and hypotenuse c, drawn straight line segments from the point F to the point I with a side length b, drawn the line segment perpendicular to AB, from the point F to point I to the length of the side b. Pull the line segment perpendicular to the IF, from the first point to duck H and from point I to point J with a side length. Line segment drawn from point A to point H and F sides with side length c. Line segment drawn from point H to point A and from point A to point. Connect both isosceles trapezoid ACJH and CELJ to trapezoidal AELH, such

that trapezoidal  $ACJH \cong CELJ$ , then come into force  $a^2 + b^2 = c^2$

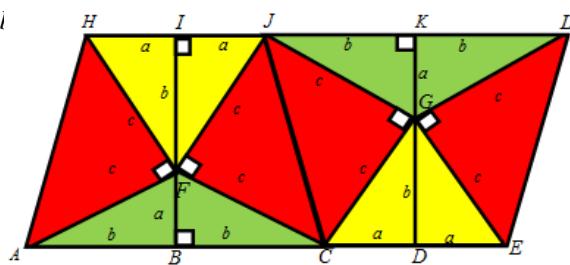


Fig.3: Parallelogram AEHL

### Proof:

In a parallelogram  $AELH$  constructed from two congruent trapezoids  $ACJH$  and  $CELJ$ . In trapezoid  $ACJH$  consists of three pairs of right-angled triangles are congruent ie  $\Delta ABF \cong \Delta CBF$  and  $\Delta HIF \cong \Delta JIF$  which has a high pedestal of  $a$  and  $b$ , and  $\Delta AFH \cong \Delta CFJ$  which has a high pedestal and  $c$ , such a parallelogram  $AELH$  has eight right-angled triangles are congruent with the high pedestal of  $a$  and  $b$ , and four congruent right-angled triangle with the base and the height is  $c$ , then come into force  $a^2 + b^2 = c^2$ .

$$\text{Area } AELH = 4 \times \text{Area } FHI + 4 \times \text{Area } AFH + 4 \times \text{Area } ABF$$

$$2\left(\frac{1}{2}(2a+2b)(a+b)\right) = 4\frac{1}{2}ab + 4\frac{1}{2}c^2 + 4\frac{1}{2}ab$$

$$2a^2 + 2ab + 2ab + 2b^2 = 2ab + 2c^2 + 2ab$$

$$2a^2 + 4ab - 4ab + 2b^2 = 2c^2$$

$$2a^2 + 2b^2 = 2c^2$$

$$a^2 + b^2 = c^2$$

**Theorem 4:** a trapezoid  $ABIH$  constructed of  $ABCD$  square and  $DCIH$  trapezoid. On the square  $ABCD$  constructed from  $EJ$  right triangle with the base  $a$ , height  $b$  and hypotenuse  $c$ , drawn straight line segment from point  $J$  to point  $C$  to length  $b$  and from point  $E$  to point  $A$  with a side length. Pull the line segment perpendicular to  $AB$  from point  $A$  to point  $F$  with the length  $b$ , connect the line segment from point  $E$  to point  $F$ . Pull the line segment perpendicular to  $BC$  from point  $C$  to point  $G$  with length  $a$  and from the point  $G$  to point  $D$  with a length  $b$ . Pull the line segment with a length  $c$  from point  $J$  to point  $G$  and point  $F$  to point  $G$ , draw a line segment perpendicular to  $BC$ , from point  $D$  to point  $H$  and from point  $C$  to point  $I$ . Connect the line segment from point  $H$  and  $I$  to point  $G$ , and,  $I$  point to point  $H$ , such that  $ABHF \cong HCDE DCIH$  then come to form  $a^2 + b^2 = c^2$ .

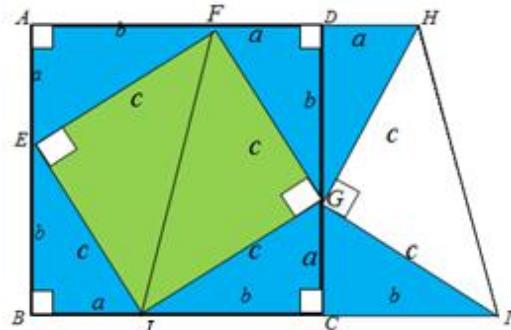


Fig.4: Trapezoid ABIH

### Proof:

A square  $ABCD$  constructed of four right-angled triangles with a pedestal length and height  $b$ , and two right-angled triangles with a base and height  $c$ . In the wake square  $ABCD$ , drawn a line segment from point  $D$  to point  $H$  to the base  $a$ , and drawn a line segment from point  $C$  to point  $I$  with a side length  $b$  and retractable segment line from point  $G$  cut  $CD$  at point  $H$  and  $I$  each had side length  $c$ . Line segment drawn from point  $H$  to the point  $I$  with the length  $\sqrt{2}c$ , so that  $\Delta EJB \cong \Delta EAF \cong \Delta EDG \cong \Delta JCG \cong \Delta HDG \cong \Delta ICG$  and  $\Delta FEH \cong \Delta FGH \cong \Delta HGI$ , then apply  $a^2 + b^2 = c^2$ .

$$\text{Area of rectangle } ABCD + \text{Area of rectangle } CDHI = 6. \text{area triangle Area } AEF + 3 \text{ Area of triangle } EFH$$

$$(a+b)(a+b) + \frac{1}{2}(a+b)(a+b) = 6\left(\frac{1}{2}ab\right) + 3\left(\frac{1}{2}c^2\right)$$

$$\frac{2a^2 + 4ab + 2b^2 + a^2 + 2ab + b^2}{2} = \frac{6ab + 3c^2}{2}$$

$$3a^2 + 6ab + 3b^2 = 6ab + 3c^2$$

$$3a^2 + 3b^2 = 3c^2$$

$$a^2 + b^2 = c^2$$

Or,

$$\text{Area } ABIH = 6. \text{Area } AEF + 3. \text{Area } EFH$$

$$\frac{1}{2}(a+2b)(2a+b)(a+b) = 6\left(\frac{1}{2}ab\right) + 3\left(\frac{1}{2}c^2\right)$$

$$\frac{3a}{2} = \frac{6ab + 3c^2}{2}$$

$$3a^2 + 6ab + 3b^2 = 6ab + 3c^2$$

$$3a^2 + 3b^2 = 3c^2$$

$$a^2 + b^2 = c^2$$

**Theorem 5:** an  $ABDE$  rectangle is constructed from two square pieces  $ABCF$  and  $CDEF$ . Each square constructed of four right-angled triangles are congruent with the base  $a$  and height  $b$ , and a square  $PQRS$  with

sides  $(b-a)$ , drawn line segment from point  $F$  and point  $C$ , such that  $ABDE$  form a rectangle with a length  $2c$  and width  $c$ , then apply  $a^2 + b^2 = c^2$ .

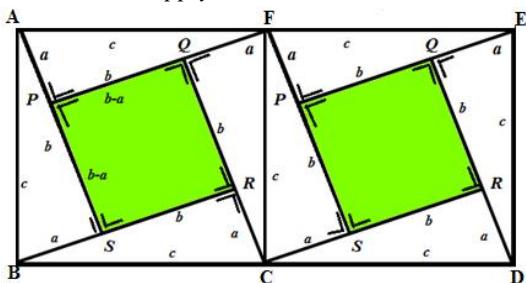


Fig.5 : Rectangle ABDE

#### Proof:

A rectangle  $a b d e$  of two square pieces is congruent.  $ABDC$  square with sides  $c$ .  $ABDC$  square have the four right-angled triangles are congruent with the high pedestal of  $a$  and  $b$ , and square  $PQRS$  with sides  $(b-a)$ . Such that  $AP = BS = RC = FQ = a$ , and  $AS = CQ = FP = BR = b$ , then apply  $a^2 + b^2 = c^2$ .

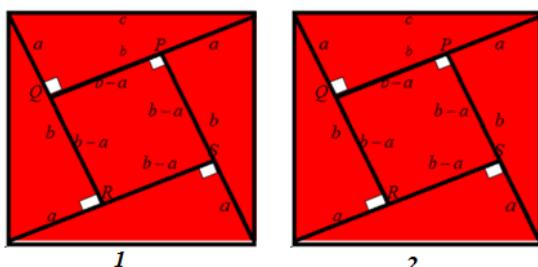


Fig.6 : Rectangular ABCD

To prove the Pythagorean theorem above note the following:

Area Rectangle  $= 2.(4.$  Area of triangle ARB + Area of rectangle PQRS $)$

$$p.l = 2 \left( 4 \cdot \frac{1}{2} ab + (b-a)^2 \right)$$

$$2c \cdot 2c = 2 \left( 2ab + b^2 - 2ab + a^2 \right)$$

$$2c^2 = 2(b^2 + a^2)$$

$$2c^2 = 2a^2 + 2b^2$$

$$c^2 = a^2 + b^2$$

### III. CONCLUSION

The above discussion has discovered the five new ways of proving Pythagoras Theorem. The fifth way is quite effective in a very famous proving and rewarding theorem. However, the five new evidences above are pretty easy to be understood by teachers and students in the school.

### ACKNOWLEDGEMENTS

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