An Application of Distributional Two Dimensional Fourier-Mellin Transform

V. D. Sharma[1], P. D. Dolas[2]

[1]HOD, Department of Mathematics, Arts, Commerce & Science College, Kiran Nagar, Amravati, Maharashtra, India
[2]Department of Mathematics, Dr. Rajendra Gode Institute of Technology & Research, Amravati, Maharashtra, India.

Abstract— *Integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. They provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. With ever greater demand for mathematical methods to provide a both theory and applications for science and engineering, the utility and interest of integral transforms seems more clearly established than ever. In spite of the fact that integral transforms have many mathematical and physical applications, their use is still predominant in advanced study and research.*

Keeping these features in mind in this paper we provide the solution of differential equation for the distributional two dimensional Fourier-Mellin transform of the type $P\left(\wedge_{t,l,x,y}^{*}\right) u(t,l,x,y) = f(t,l,x,y)$ *and* $P\left(D_{t,l,x,y}\right)u(t,l,x,y) = f(t,l,x,y)$ *using the differential operator* $\wedge_{t,l,x,y}$ *and* $\wedge_{t,l,x,y}^{*}$.

Keywords— ***Adjoint operator, Fourier Transform, Generalized function, Mellin Transform, Two Dimensional Fourier-Mellin Transform.***

I. INTRODUCTION

Characteristically, one uses the integral transformation as mathematical or physical tool to alter the problems into one that can be solved. Integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The concept of integral transform originated from Fourier integral formula. It is composed of four objects & defined as

$$F(s) = \int_{t_1}^{t_2} f(t)\ K(t,s)\, dt$$

There are two functions ($F(s)$ as output, $f(t)$ as input) that changes depending on problem and two objects which are remain same for a transform irrespective of input function ($K(t,s)$ as kernel, interval). Kernels are different for different type of transforms. They operate on function in much the same way as (certain) matrices do on vectors [2]. Integral Transform methods providing unifying mathematical approach to the study of electrical, network, devices for energy conversion and control, antennas and other component of electrical system [3]. Integral transforms transform a class of differential equations into a class of algebraic equations; the algebraic equations can be solved easily. So, we can easily find the solutions of those differential equations, this is the most useful significance of integral transforms. These are also providing powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations [1]. These are equally applied to the subject of electrical communication by wire or optical fibers, to wireless radio propagation [3]. This is a key point because it provides a way to understand exactly what an integral transform does. The integral transforms provide both theory and applications for science and engineering. Integral Transform methods are at the heart of engineering curriculum. In spite all that things their uses are still predominant in advanced study and research. So, these integral transforms are attracting to all researchers, since, their fabulous work and applications in almost all fields. In the present paper, we have focused on applications of Distributional Two Dimensional Fourier-Mellin Transform to differential equations. For this purpose, we have putting the following definitions which are already defined in our previous papers [4,7,8,9] as-

1.1. The Test Function space $FM_{a,b,c,d,\alpha}$

Let I be the open set in $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, the space $FM_{a,b,c,d,\alpha}$ is given by

$$FM_{a,b,c,d,\alpha} = \{\emptyset : \emptyset \in E_+ / \gamma_{a,b,c,d,k,r,h g.q,j}\emptyset(t,l,x,y) = \sup_{I_1}\left|t^k l^r \xi_{a,b}(x) x^{h+1} \eta_{c,d}(y) y^{g+1} D_t^q D_x^h D_l^j D_y^g \emptyset(t,l,x,y)\right| < C\, A^k\, k^{k\alpha} \qquad k,r,h,g,g,q,j = 0,1,2,3 \ldots\ldots\ldots$$

(1.1.1)

Where, the constant C, A depending on testing function space $\emptyset$.

1.2. Two Dimensional Distributional Fourier-Mellin Transform of Generalized function in $FM_{a,b,c,d,\alpha}^{*}$

The Two Dimensional Distributional Fourier-Mellin Transform is defined as-

$$FM\{f(t,l,x,y)\} = F(s,u,p,v) = \langle f(t,l,x,y), e^{-i(st+ul)} x^{p-1} y^{v-1}\rangle \qquad (1.2.1)$$

https://dx.doi.org/10.22161/ijaers/3.11.31 **ISSN: 2349-6495(P) | 2456-1908(O)**

Where for each fixed $x(0<x<\infty), y(0<y<\infty), t(0<t<\infty)$ and $l(0<l<\infty)$.

The R.H.S. of (2.1) has sense of an application of $f(t,l,x,y) \in FM^*_{a,b,c,d,\alpha}$ to $e^{-i(st+ul)}x^{p-1}y^{v-1} \in FM_{a,b,c,d,\alpha}$.

In the present paper, we have defined the differential operator $\Lambda_{t,l,x,y}$, its adjoint operator $\Lambda^*_{t,l,x,y}$. Using this differential operator in the given paper we have present an application of Two Dimensional Fourier-Mellin Transform to differential equations. The notation and terminology are given as per A.H. Zemanian [5, 6].

II. AN APPLICATION OF DISTRIBUTIONAL TWO DIMENSIONAL FOURIER-MELLIN TRANSFORM

The kernel of Distribution Two Dimensional Fourier-Mellin Transform

$$K(t,l,x,y,s,u,p,v) = e^{-i(st+ul)}x^{p-1}y^{v-1} = e^{-ist}e^{-iul}x^{p-1}y^{v-1}$$

Now differentiate above term w.r.t. t, l, x and y, we get

$D_tD_lD_xD_y\, K(t,l,x,y,s,u,p,v) = (-is)e^{-ist}(-iu)e^{-iul}(p-1)x^{p-2}(v-1)y^{v-2} = (is)(iu)(p-1)(v-1)e^{-ist}e^{-iul}x^{p-2}y^{v-2}$

$D_tD_lD_xD_y\, K(t,l,x,y,s,u,p,v) = -su(p-1)(v-1)e^{-ist}e^{-iul}x^{p-2}y^{v-2}$ Now multiplying both side by xy, we get

$$xyD_tD_lD_xD_y\, K(t,l,x,y,s,u,p,v) = -su(p-1)(v-1)e^{-ist}e^{-iul}x^{p-1}y^{v-1} = -su(p-1)(v-1)K(t,l,x,y,s,u,p,v)$$

Now we construct an operator

$\Lambda_{t,l,x,y} = xyD_tD_lD_xD_y - su(p-1)(v-1)$

Where, $D_t = \frac{d}{dt}, D_l = \frac{d}{dl}, D_x = \frac{d}{dx}, D_y = \frac{d}{dy}$

$\Lambda_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = [xyD_tD_lD_xD_y - su(p-1)(v-1)]K(t,l,x,y,s,u,p,v) = xyD_tD_lD_xD_yK(t,l,x,y,s,u,p,v) - su(p-1)(v-1)K(t,l,x,y,s,u,p,v) = -su(p-1)(v-1)K(t,l,x,y,s,u,p,v) - su(p-1)(v-1)K(t,l,x,y,s,u,p,v)\Lambda_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = -2su(p-1)(v-1)K(t,l,x,y,s,u,p,v)$ Now Consider-

$\Lambda_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (C_0)\, K(t,l,x,y,s,u,p,v)$

Where, $C_0 = -2su(p-1)(v-1)$

Continuing in this way we get

$\Lambda^2_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (C_0)^2 K(t,l,x,y,s,u,p,v)$

$\Lambda^3_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (C_0)^3 K(t,l,x,y,s,u,p,v)$

$\Lambda^4_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (C_0)^4 K(t,l,x,y,s,u,p,v)$

………… and so on

$\Lambda^k_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (C_0)^k K(t,l,x,y,s,u,p,v) = (-2su(p-1)(v-1))^k K(t,l,x,y,s,u,p,v)$

Since, the operator $\Lambda^k_{t,l,x,y}\, K(t,l,x,y,s,u,p,v) = (-2su(p-1)(v-1))^k K(t,l,x,y,s,u,p,v)$ is obviously linear and continuous, we have

$FM\{\Lambda^k_{t,l,x,y}\, K(t,l,x,y,s,u,p,v)\} = \langle \Lambda^k_{t,l,x,y}\, f(t,l,x,y), K(t,l,x,y,s,u,p,v)\rangle = \langle f(t,l,x,y), \Lambda^k_{t,l,x,y}K(t,l,x,y,s,u,p,v)\rangle FM\{\Lambda^k_{t,l,x,y}\, K(t,l,x,y,s,u,p$ $\langle f(t,l,x,y), (-2su(p-1)(v-1))^k\, K(t,l,x,y,s,u,p,v)\rangle$ For all $f \in FM^*_{a,b,c,d,\alpha}$.

III. ADJOINT OPERATOR

We have define an operator $\Lambda^*_{tl.,x,y}: FM^*_{a,b,c,d,\alpha} \to FM_{a,b,c,d,\alpha}$ using the relation

$\langle \Lambda^*_{t,l,x,y}\, f(t,l,x,y), \emptyset(t,l,x,y)\rangle = \langle f(t,l,x,y), \Lambda_{t,l,x,y}\emptyset(t,l,x,y)\rangle$ For all $f \in FM^*_{a,b,c,d,\alpha}$ and $\emptyset \in FM_{a,b,c,d,\alpha}$. The operator $\Lambda^*_{t,l,x,y}$ is called the adjoint operator of $\Lambda_{t,l,x,y}$. For each $k = 1,2,3,\ldots\ldots\ldots$ we can easily get-$\langle (\Lambda^*_{t,l,x,y})^k\, f(t,l,x,y), \emptyset(t,l,x,y)\rangle = \langle f(t,l,x,y), \Lambda^k_{t,l,x,y}\emptyset(t,l,x,y)\rangle$ It can be easily shown that if f is regular distribution generated by an element in $FM_{a,b,c,d,\alpha}$. Then, $\Lambda^*_{t,l,x,y}\, f = \Lambda_{t,l,x,y}\, f$,

For each $k = 1,2,3 \ldots\ldots\ldots$

We have-$\langle (\Lambda^*_{t,l,x,y})^k\, f(t,l,x,y), K(t,l,x,y,s,u,p,v)\rangle = \langle f(t,l,x,y), \Lambda^k_{t,l,x,y}K(t,l,x,y,s,u,p,v)\rangle = \langle f(t,l,x,y), (-2su(p-1)(v-1))^k\, K(t,l,x,y,s,u,p,v)\rangle = \langle f(t,l,x,y), (C_0)^k\, K(t,l,x,y,s,u,p,v)\rangle$

$\langle (\Lambda^*_{t,l,x,y})^k\, f(t,l,x,y), K(t,l,x,y,s,u,p,v)\rangle = (C_0)^k \langle f(t,l,x,y), K(t,l,x,y,s,u,p,v)\rangle FM\left\{(\Lambda^*_{t,l,x,y})^k\, f(t,l,x,y)\right\}$

$(C_0)^k\, FM\{f(t,l,x,y)\}$ For all $f \in FM^*_{a,b,c,d,\alpha}$.

IV. AN APPLICATION OF THE TWO DIMENSIONAL FOURIER-MELLIN TRANSFORM TO DIFFERENTIAL EQUATION

4.1 SOLUTION OF $P(\Lambda^*_{t,l,x,y})\, u(t,l,x,y) = f(t,l,x,y)$

Consider the Differential equation as

$$P(\Lambda^*_{t,l,x,y})\, u(t,l,x,y) = f(t,l,x,y) \qquad (4.1.1)$$

Where, $f \in FM^*_{a,b,c,d,\alpha}$ and P is any polynomial of degree m. Suppose that the equation (4.1.1) possesses the solution u. Applying Two Dimensional Fourier-Mellin Transform to (4.1.1) we get-

$FM\{P(\Lambda^*_{t,l,x,y})\, u\} = FM\{f\}\, P(-2su(p-1)(v-1))FM\{u(t,l,x,y)\} = FM\{f(t,l,x,y)\}$ {Since by using $\left\{(\Lambda^*_{t,l,x,y})^k\, f(t,l,x,y)\right\} = [-2su(p-1)(v-1)]^k\, FM\{f(t,l,x,y)\}$

https://dx.doi.org/10.22161/ijaers/3.11.31 *ISSN: 2349-6495(P) | 2456-1908(O)*

$P(C_0)FM\{u\} = FM\{f\}$ (4.1.2)

If we further assume that the polynomial P is such that for $\in > 0$. $|P(C_0)| < \in \neq 0$

For $s > 0, u > 0, p > 0, v > 0$ (4.1.3)

Then under this assumption (4.1.2) gives-

$FM(u) = [P(-2su(p-1)(v-1))]^{-1}\, FM(f)$ (4.1.4)

Applying the inversion of Two Dimensional Fourier-Mellin Transform we get

$$u = FM^{-1}\left\{\frac{FM(f)}{P[-2su(p-1)(v-1)]}\right\}$$

$$u = FM^{-1}\left\{\frac{FM(f)}{P(C_0)}\right\} \quad (4.1.5)$$

4.2 SOLUTION OF DIFFERENTIAL EQUATION

$$P(D_{t,l,x,y})u(t,l,x,y) = f(t,l,x,y)$$

Consider the differential equation

$P(D_{t,l,x,y})u(t,l,x,y) = f(t,l,x,y)$ (4.2.1)

When $f \in FM^*_{a,b,c,d,\alpha}$ and $P(D_{t,l,x,y}) = \sum_{\substack{|\alpha|\le m \\ |\beta|\le n \\ |\gamma|\le m' \\ |\delta|\le n'}} a_\alpha D_t^\alpha a_\beta D_l^\beta a_\gamma D_x^\gamma a_\delta D_y^\delta$ is a linear differential operator of order m, n, m', n' with constant coefficients $a_\alpha, a_\beta, a_\gamma, a_\delta$ respectively. Suppose that the equation (4.2.1) possesses a solution u. Applying Two Dimensional Fourier-Mellin Transform to (4.2.1) and using we get

$xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v) =$
$(-is)^q(-iu)^q(p-1)^j(v-1)^g e^{-i(st+ul)}x^{p-1}y^{v-1} =$
$(-is)^q(-iu)^q(p-1)^j(v-$
$1)^g K(t,l,x,y,s,u,p,v) xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v) =$
$(-1)^{q+h}\,(is)^q(iu)^q(p-1)^j(v-$
$1)^g K(t,l,x,y,s,u,p,v)$ We have,

$FM\{P(D_{t,l,x,y})u(t,l,x,y)\} = FM\{f(t,l,x,y)\}$ (4.2.2)

We can reform them to the Two Dimensional Fourier-Mellin Transform and hence we get

$P[xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v)]\, FM(u) = FM(f)$ (4.2.3)

Under the assumption that the polynomial P is such that

$P[xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v)] < \epsilon$

for $\epsilon > 0 \in R^n$

Using (4.2.3)

$FM(u) =$
$P[xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v)]^{-1}\, FM(f)$Applying the inversion of Two Dimensional Fourier-Mellin Transform to above equation, we have

$$u = [FM]^{-1}\left\{\frac{FM(f)}{P[xy\, D_t^q D_l^h D_x^j D_y^g\, K(t,l,x,y,s,u,p,v)]}\right\}$$

V. CONCLUSION

In this paper we have given an application of Distributional Two Dimensional Fourier-Mellin transform to differential equation by introducing new operator of $\Lambda_{t,l,x,y}$ and its adjoint operator$\Lambda^*_{t,l,x,y}$.

REFERENCES

[1] Lokenath Debnath and Dambaru Bhatta, Integral Transforms and their Applications, Chapman and Hall/CRC Taylor and Francis Group Boca Raton London, New York, 2007.

[2] J. Berian James: The Integral Transform for you & me, Royal Observatory Edinburgh Institute for Astronomy, March 2008.

[3] Ronald N. Bracewell: The Fourier Transform and its Applications, Third Edition, McGraw-Hill International Edition, 2000.

[4] V.D. Sharma, P.D. Dolas: Inversion formula for Two Dimensional Generalized Fourier-Mellin Transform & its applications, Int. Jr. of Advanced Scientific and Technical Research, December 2011, Issue 1, Vol. 2.

[5] A. H. Zemanian, “Generalized integral t ransform,” Inter science publisher, New York, 1968.

[6] A. H. Zemanian, “Distribut ion theory and t ransform analysis,” McGraw Hill, New York, 1965.

[7] V.D. Sharma, P.D. Dolas: Convolution Theorem for Two Dimensional Fourier-Mellin Transform, International Journal of Research in Engineering and Applied Sciences, Vol. 6, Issue 3 (March, 2016).

[8] V.D. Sharma, P.D. Dolas: Representation Theorem For The Distributional Two Dimensional Fourier-Mellin Transform, International Journal of Mathematical Archive-5(9), 2014, 258-262.

[9] V.D. Sharma, A. N. Rangari: An application of Generalized Fourier-Finite Mellin Transform, International Journal of Innovative Research in Science, Engineering and Technology, Vol. 5, Issue 1, January 2016.

[10] V.D. Sharma, P.D. Dolas: Adjoint Operators of Two Dimensional Fourier-Mellin Transform, International Journal Of Pure And Applied Research In Engineering And Technology, 2016; Volume 4 (9): 64-70.