GROUP CLASSIFICATION OF LINEAR SECOND-ORDER DELAY ORDINARY DIFFERENTIAL EQUATION

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Abstract. *The linear delay ordinary differential equation*

$$y''(x)+a(x)y'(x)+b(x)y'(x-\tau)+c(x)y(x)+d(x)y(x-\tau)=g(x)$$

is studied, where the coefficients $a(x), b(x), c(x)$ *and* $d(x)$ *and function* $g(x)$ *are arbitrary. In this manuscript, group analysis is applied to find equivalent symmetries of the equation.*

1 Introduction

Let us consider a linear second-order delay ordinary differential equation

$$y''(x)+a(x)y'(x)+b(x)y'(x-\tau)+c(x)y(x)+d(x)y(x-\tau)=g(x). \tag{1}$$

For brevity, the symbol y_τ will be used to denote $y(x-\tau)$, y to denote $y(x)$ and y', y_τ' will mean the first derivatives of y at point x and $x-\tau$, respectively. Then equation (1) can be simply written as

$$y''+a(x)y'+b(x)y_\tau'+c(x)y+d(x)y_\tau=g(x). \tag{2}$$

Here it is assumed that $b^2+c^2\neq 0$. Application of this equation can be found in biology, physics and engineering, where it is used to model natural phenomena.

2 Lie Group of Transformations

In [6], a definition of admitted Lie group of transformations for delay differential equations was developed. A theory of equivalence Lie groups can be considered similar to that of admitted Lie groups. Here this analysis is developed.

Let $\varphi:\Omega\times\Delta\to\Omega$ be a transformation, where Ω is a set of variables $(x,y,\phi), \phi=(a,b,c,d,g)$ and $\Delta\subset R$ is a symmetric interval with respect to zero. The variable $\varepsilon\in\Delta$ is considered as a parameter of the transformation φ. This transform maps the variables (x,y,ϕ) to variables $(\bar{x},\bar{y},\bar{\phi})$. Let $\varphi(x,y,\phi;\varepsilon)$ be denoted by $\varphi_\varepsilon(x,y,\phi)$. The set of functions φ_ε forms *a one-parameter transformation Lie group* of the space Ω if it contains the identity transformation as well as inverse of its elements and their composition [4,5,6]. Alternatively, the notation $\bar{x}=\varphi^x(x,y,\phi;\varepsilon), \bar{y}=\varphi^y(x,y,\phi;\varepsilon), \bar{\phi}=\varphi^\phi(x,y,\phi;\varepsilon)$ is used instead of $\varphi_\varepsilon=(\bar{x},\bar{y},\bar{\phi})$. The transformed variable y with delay term, it's derivatives and it's derivatives with delay term are defined by $\bar{y}_\tau=\bar{y}(\bar{x}-\tau)$, $\bar{y}'=\dfrac{d\bar{y}}{d\bar{x}}$ and $\bar{y}_\tau'=\dfrac{d\bar{y}}{d\bar{x}}(\bar{x}-\tau)$, respectively.

Let us consider a delay differential equation

$$F(x,y,y_\tau,y',y_\tau',y'',\phi)=0. \tag{3}$$

A Lie group of transformations is called *admitted* if each transformation maps a solution of the differential equation to a solution of the same equation. Such transformation are called *symmetries*. For this reason, equations for defining symmetries were constructed under the assumption that a Lie group of transformations maps a solution of a delay differential equation into a solution of the same equation. This assumption leads to

$$\frac{\partial F(\bar{x},\bar{y},\bar{y}_\tau,\bar{y}',\bar{y}_\tau',\bar{y}'',\bar{\phi})}{\partial\varepsilon}\Big|_{\varepsilon=0} = \tilde{X}F(x,y,y_\tau,y',y_\tau',y'',\phi)\big|_{(1.2)} = 0. \tag{4}$$

Here, the operator $\tilde{X}$ is defined by

$$\tilde{X} = (\eta - y'\xi)\partial_y + (\eta_\tau - y_\tau'\xi_\tau)\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y_\tau'}\partial_{y_\tau'} + \eta^{y''}\partial_{y''} + (\zeta - \phi_x\xi)\partial_\phi + \zeta^{\phi_y}\partial_{\phi_y}.$$

where

$$\xi(x,y,\phi) = \frac{\partial\varphi^x}{\partial\varepsilon}(x,y,\phi;0), \quad \eta(x,y,\phi) = \frac{\partial\varphi^y}{\partial\varepsilon}(x,y,\phi;0),$$

$$\zeta(x,y,\phi) = \frac{\partial\varphi^\phi}{\partial\varepsilon}(x,y,\phi;0), \quad \xi_\tau(x,y,\phi) = \xi(x-\tau,y_\tau,\phi_\tau),$$

$$\eta^{y'} = D_x(\eta - y'\xi), \quad \eta^{y''} = D_x(\eta^{y'}),$$

$$\eta_\tau(x,y,\phi) = \eta(x-\tau,y_\tau,\phi_\tau), \eta^{y_\tau'} = D_x(\eta_\tau - y_\tau'\xi_\tau), \zeta^{\phi_y} = D_y(\zeta - \phi_x\xi),$$

$$\begin{aligned} D_x &= \partial_x + y'\partial_y + y''\partial_{y'} + \ldots + y_\tau'\partial_{y_\tau} \\ &\quad +(\phi_x+\phi_y y')\partial_\phi + (\phi_{xx}+\phi_{xy}y')\partial_{\phi_x}, \\ D_y &= \partial_y + \phi_y\partial_\phi + \phi_{xy}\partial_{\phi_x} + \ldots \end{aligned}$$

The operator $\tilde{X}$ is called *a canonical Lie-Bäcklund infinitesimal operator* of a symmetry. Lie's theory [4,5,6] shows that there is a one-to-one correspondence between generators and symmetries. This operator is also equivalent to *an infinitesimal generator* [6]

$$X = \xi\partial_x + \eta\partial_y + \zeta\partial_\phi.$$

Equation (4) gives a definition of an action of the infinitesimal generator of a Lie group onto a delay differential equation.

1. Determining Equations

Definition 1. A Lie group of transformations is called *admitted* if it satisfies the equation

$$\tilde{X}F(x,y,y_\tau,y',y_\tau',y'',\phi)\big|_{(2.1)} = 0. \tag{5}$$

for any solution of (3).

Equation (5) is called *a determining equation*. For solving the determining equation one can use the theory of existence of a solution of an initial value problem for delay equation (2) [1]. This problem is formulated as follows. Let a function $\chi(x), x\in(x_0-\tau,x_0)$ be given. Find a solution $y(x), x\in[x_0,x_0+\tilde{\varepsilon})$ which satisfies the condition

$$y(x) = \chi(x),\ x\in(x_0-\tau,x_0).$$

Because the initial values are arbitrary, the variables y, y_τ and their derivatives can be considered as arbitrary elements. Thus, if the determining equation (5) is written as a polynomial of of variables and their derivatives, the coefficients of these variables in the equations must vanish. The method for obtaining the overdetermined system of equations is called *splitting the determining equation*. This gives an overdetermined system of partial differential equations for the coefficients of the infinitesimal generator. The unknown functions ξ,η and ζ can be found by solving this system.

2. Equivalence Problem

The problem of finding all equations, which are equivalent to a given equation is called *an equivalence problem*. If the given equation is a *linear* equation, then the equivalence problem is called *a linearization problem*. In this section the importance of an equivalence Lie group of transformations is shown. Consider a linear second order ordinary differential equation

$$y'' + a(x)y' + c(x)y = g(x). \tag{6}$$

A Lie group of transformations of the independent variables, dependent variables and coefficients, which conserves the differential structure of the equation is called *an equivalent Lie group*. This group allows simplifying the coefficients of equations. For example, S. Lie showed that any linear second order ordinary differential equation (6) is equivalent to the equation

$$y'' = 0. \tag{7}$$

Equation (7) admits the eight-dimensional Lie algebra spanned by the generators

$$X_1 = \frac{\partial}{\partial_x}, X_2 = \frac{\partial}{\partial_y}, X_3 = x\frac{\partial}{\partial_x}, X_4 = y\frac{\partial}{\partial_x}, X_5 = x\frac{\partial}{\partial_y},$$
$$X_6 = y\frac{\partial}{\partial_y}, X_7 = x^2\frac{\partial}{\partial_x} + xy\frac{\partial}{\partial_y}, X_8 = xy\frac{\partial}{\partial_x} + y^2\frac{\partial}{\partial_y}.$$

If one tries to find an admitted Lie group for equation (6), then the system of determining equations consists of four second-order ordinary differential equations. In general, this system cannot be solved. The purpose of this manuscript is to do group classification of equation (2).

3 Equivalence Symmetries of (2)

Let y_p be a particular solution of equation (2). Considering the change $\tilde{x} = x, \tilde{y} = y - y_p$, equation (2) is reduced to the equation

$$y'' + a(x)y' + b(x)y_\tau' + c(x)y + d(x)y_\tau = 0.$$

Similar to a second-order ordinary differential equation the coefficient $a(x)$ can be reduced by change $y = v(x)q(x)$ with q satisfying the equation $2q' + aq = 0$. We will consider equivalence symmetries of equation

$$y'' + b(x)y_\tau' + c(x)y + d(x)y_\tau = 0. \tag{8}$$

instead of (2). Letting $F = y'' + b(x)y'_\tau + c(x)y + d(x)y_\tau$, then equation (5) becomes

$$\tilde{X}\left(y'' + b(x)y_\tau' + c(x)y + d(x)y_\tau\right)|_{(3.1)} = 0. \tag{9}$$

Splitting this equation with respect to y', $y_\tau', y'_{2\tau}, b'', c'', d''$, b', c', d' and later with respect to y, y_τ, one finds

$$\xi = \xi_\tau = \alpha, \eta = \beta y + \gamma, \zeta^b = b(-\alpha' + \beta - \beta_\tau),$$
$$\zeta^c = -\beta'' - 2c\alpha', \zeta^d = -b\beta_\tau' - 2d\alpha' + d\alpha - d\alpha_\tau,$$

where $\alpha(x)$, $\beta(x)$ are arbitrary periodic functions with period τ, and $\gamma(x)$ is an arbitrary solution of (3). Thus the equivalence symmetry is

$$\overline{x} = \alpha(x),\ \overline{y} = \beta(x)y + \gamma(x). \tag{10}$$

4 Group Classification of Equation (3)

A Lie group of transformations admitted by equation (3) has to satisfy the following determining equation [3]

$$\tilde{Y}(y'' + b(x)y_\tau' + c(x)y + d(x)y_\tau)|_{(3.1)} = 0. \tag{11}$$

Here, the operator $\bar{Y}$ is defined by

$$\tilde{Y} = (\eta - y'\xi)\partial_y + (\eta_\tau - y_\tau'\xi_\tau)\partial_{y_\tau} + \eta^{y'}\partial_{y'} + \eta^{y_\tau'}\partial_{y_\tau'} + \eta^{y''}\partial_{y''}.$$

where

$$\xi(x,y) = \frac{\partial\varphi^4}{\partial\varepsilon}(x, y; 0), \quad \eta(x,y) = \frac{\partial\varphi^y}{\partial\varepsilon}(x, y; 0),$$
$$\xi_\tau(x, y) = \xi(x - \tau, y_\tau), \quad \eta_\tau(x, y) = \eta(x - \tau, y_\tau),$$
$$\eta^{y'} = D_x(\eta - y'\xi), \quad \eta^{y_\tau'} = D_x(\eta_\tau - y_\tau'\xi_\tau), \quad \eta^{y''} = D_x(\eta^{y'}),$$
$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots + y_\tau'\partial_{y_\tau} + y_\tau''\partial_{y_\tau'}.$$

Splitting this equation with respect to $y', y_\tau', y_{2\tau}'$ and later with respect to y, y_τ, one finds

$$\xi = \xi_\tau, \quad \eta = \beta y + \gamma, \tag{12}$$
$$\xi_{xx} = 2\beta', \quad \beta'' = -c'\xi - 2c\xi_x, \tag{13}$$
$$\gamma'' = -b\gamma_\tau' - c\gamma - d\gamma_\tau, \tag{14}$$
$$b(\beta - \beta_\tau) = b'\xi + \xi_x b, \tag{15}$$
$$d(\beta - \beta_\tau) = d'\xi + b\beta_\tau' + 2\xi_x d. \tag{16}$$

By integrating (13), one finds $\beta = \xi_x / 2 + C_1$, where C_1 is an arbitrary constant. Since $\xi = \xi_\tau$, it implies $\beta = \beta_\tau$. Hence, integrating equation (15) one has

$$b\xi = C_2, \tag{17}$$

C_2 is an arbitrary constant. Equation (16) is written as

$$d'\xi + 2\xi_x d = -\frac{b}{2}\xi_{xx}. \tag{18}$$

1. Case $b \neq 0, d \neq 0$

Substituting β into the second equation (13), one gets

$$\xi\xi_{xx} - \frac{\xi_x^2}{2} + 2c\xi^2 = C_3, \tag{19}$$

C_3 is an arbitrary constant. If $C_2 \neq 0$, then from equations (17), (18) and (19), one obtains

$$\xi = \frac{C_2}{b}, \eta = y\left(\frac{C_2}{2b'} + C_1\right) + \gamma,$$
$$c = \frac{1}{2}\left[C_5 b^2 - \frac{3}{2}\left(\frac{b'}{b}\right)^2 + \frac{b''}{2b}\right], d = \frac{b'}{2} + C_4 b^2,$$

where C_4 is an arbitrary constants, $C_5 = C_3/C_2$, and $\gamma(x)$ is an arbitrary solution of (3). Since $\xi = \xi_\tau$, the the coefficient b has to satisfy the same property $b = b_\tau$. The infinitesimal generator obtained is

$$X = C_1 y\partial_y + C_2\left(\frac{1}{b}\partial_x + \frac{y}{2}\left(\frac{1}{b}\right)'\partial_y\right) + \gamma\partial_y. \tag{20}$$

If $C_2 = 0$, then $\xi = 0, \eta = C_1 y + \gamma$ and all coefficients are arbitrary. The infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{21}$$

2. Case $b \neq 0, d = 0$

Solving equations (16), (17), (18) and the second equation of (13), one obtains $\beta_\tau = C_6, b\xi = C_2, \xi = C_7 x + C_8, c\xi^2 = C_9$, where C_6, C_7, C_8, C_9 are arbitrary constants. Since $\xi = \xi_\tau$, then $C_7 = 0$. If $C_8 \neq 0$, then

$$c = \frac{C_9}{C_8^2}, b = \frac{C_2}{C_8}. \tag{22}$$

The infinitesimal generator of the admitted Lie group is

$$X = C_8\partial_x + (C_6 y + \gamma)\partial_y. \tag{23}$$

If $C_8 = 0$, then $\xi = 0, \eta = C_6 y + \gamma$, b and c are arbitrary, $\gamma(x)$ is an arbitrary solution of (8). The infinitesimal generator is

$$X = (C_6 y + \gamma)\partial_y.$$

1. Case $b = 0, d \neq 0$

From equation (18) one finds $d\xi^2 = C_{10}$, where C_{10} is an arbitrary constant. Hence,

$$\xi = \left(\frac{C_{10}}{d}\right)^{1/2}, \eta = -\left(\frac{C_{10}}{4}\frac{d'}{d^{3/2}} + C_1\right)y + \gamma. \tag{24}$$

If $C_{10} \neq 0$, then from equation (19) one finds

$$c = \frac{1}{2}\left[\frac{C_3}{C_{10}}d + \frac{d'}{2d} + \frac{1}{8}(\frac{d'}{d^2})^2\right]. \tag{25}$$

The infinitesimal generator obtained is

$$X = \frac{C_{10}}{d^{1/2}}\partial_x + \left(-\frac{C_{10}^{1/2}d'}{2d^{3/2}} + C_1 + \gamma\right)\partial_y. \tag{26}$$

If $C_{10} = 0$, then $\xi = 0, \beta = C_1, \eta = C_1 y + \gamma$ and the coefficients c and d are arbitrary functions. Hence, the infinitesimal generator is

$$X = (C_1 y + \gamma)\partial_y. \tag{27}$$

5 Conclusion

The linear second-order delay ordinary differential equation is classified into three cases as the followings:

- $b \neq 0$ and $d \neq 0$. The infinitesimal generator admitted by the equation of this case is (20)
- $b \neq 0$ and $d = 0$. The infinitesimal generator admitted by the equation of this case is (23)
- $b = 0$ and $d \neq 0$. The infinitesimal generator admitted by the equation of this case is (26).

Acknowledgements

This work is supported by Ministry of University Affairs of Thailand (MUA) and Nakhon Phanom University. Author is deeply indebted to S. V. Meleshko for his frequent help.

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