SLOWLY ROTATING WORMHOLE SOLUTION

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Abstract

The ideas introduced by Einstein General Relativity theory toward the development of wormhole research field are reviewed, namely, the concept of spacetime, Riemannian geometry, cosmological constant and the pioneering work of Einstein and Rosen on the subject. Spherically symmetric spacetime metric that provides the wormhole spacetime metric by the revival works of Morris and Thorne are discussed. Concerning recent works on the subject, a generalized wormhole solution with cosmological constant introduced by the works of Lemos, Lobo and Oliveira are also discussed. Finally, the rotating wormhole solution as introduced by Khatsymovsky and Teo has provided a more generalized solution compared with the Morris-Thorne static solution. Based on these ideas, we proposed the spacetime metric of a slowly rotating spherically symmetric wormhole where we constructed set of Einstein equations in term of pressure and the energy density of the wormhole.

Keywords: *General Relativity; Cosmological constant; Einstein-Rosen Bridge; Wormhole*

1. Introduction

The possibility for the existence of wormhole has long been considered as topologically non-trivial exact solution to the Einstein field equations. Einstein field equations that are derived from general relativity theory shows the relation between space-time curvature and the energy momentum tensor that represents gravitation [1] and matter or energy respectively. The equations are nonlinear and are highly complicated to solve analytically. However, Schwarzchild had considered a special case which provides an exact solution to the equations, namely, the static spherically symmetric field produced by a spherically symmetric body at rest [2]. Later it has been shown by Einstein and Rosen [3] that the Schwarzchild solution has a singularity. The idea of singularity was then propagated toward further research on black-holes. Not satisfied with the possibility of the existence of singularity, Einstein and Rosen in their paper [3], attempted to build a geometrical model of a physical elementary "particle" that was everywhere finite and singularity-free. This idea was put forward in order to oppose some of ideas exist during the era that the material particles were considered as singularities of the field. Einstein and Rosen solutions provide the mathematical representation of physical space of two identical sheets where a particle is represented by a "bridge" connecting these sheets. This model of elementary particle however was considered a failure but it had generated the idea of Einstein-Rosen bridge and as coined later by Wheeler [4] the term "wormhole" where the ideas concerning the space-time foam, was then introduced. This was described as a microscopic charged carrying wormhole.

2. Traversable wormhole

The idea of a traversable wormhole was originally introduced by Morris and Thorne [5]. Unlike the Einstein-Rosen-bridge [3], or the Wheeler's wormhole [4], a traversable wormhole [6, 7, 8] permit the two ways travel of objects. Despite the still questionable possibilities for the existence of such wormhole, their study had generated newly exciting areas of research such as its fundamental properties, the application for faster than light travel and the associated problems of causality violation as suggested in the Hawking's paper concerning the chronology protection conjecture [9, 10].

Einstein-Rosen-bridge or the Schwarzchild wormhole discard the presence of singularity, however, it contains an event horizon, which is non-traversable because there exist constraints similar to the case of being trapped in a black-hole. Morris-Thorne revived a new idea of traversable wormhole [5]. The minimum requirement for a wormhole to be traversable is that there shall be no event horizon in the system. It is therefore a traversable wormhole space-time containing no curvature singularity. However, the problem with the traversable wormhole is that, matter near the throat of the wormhole violates the null energy condition (NEC). From classical perspective the violation of the NEC is impossible, but some quantum effects violate the NEC [11].

For mathematical convenience Morris-Thorne assumed that their traversable wormholes were time independent non-rotating and spherically symmetric bridge between two universes (inter-universe) or two distant places in the same universe (intra-universe). Both conditions, even though there are differences globally but locally their behavior near the wormhole throat are the same and they posses two asymptotically flat region. The proper radial distance is denoted as l, and based on the Schwarzchild metric [2]

$$ds^2 = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{1}$$

the space time metric as proposed by Morris -Thorne can be written in general as

$$ds^2 = -e^{2\Phi(l)} c^2 dt^2 + dl^2 + r^2(l)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2}$$

where $l \in (-\infty, +\infty)$. The two asymptotically flat regions occur at $l \approx \pm\infty$ and the radius of the wormhole throat is defined by $r_0 = \min\{r(l)\}$. Morris-Thorne then reduced the functional dependency of the metric, to obtain a simpler expression for further calculation of Reimann, Ricci and Einstein tensors. The Morris-Thorne expression in Schwarzchild (ct, r, θ, φ) coordinates [5] then becomes

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{3}$$

where $\Phi(r)$ is called the red shift function, while $b(r)$ is called the shape function. The choice of the shape function was based on the embedding diagram (Fig.1) that is used to represent a wormhole spatial geometry, which result in the equation of the embedding surface [5]

$$\frac{dz}{dr} = \left(\frac{r}{b} - 1\right)^{-\frac{1}{2}} \tag{4}$$

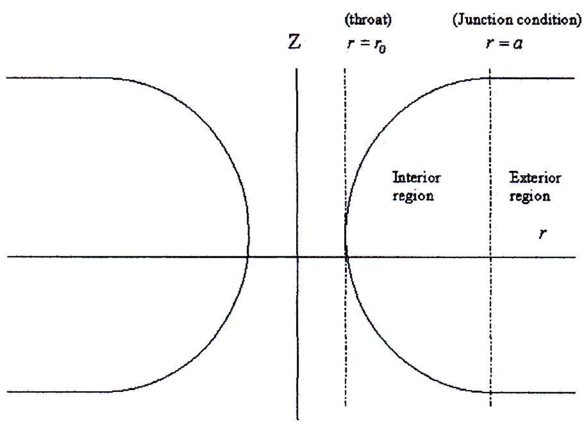

Figure1; The profile of the embedding diagram

This equation (4) displays the manner in which the function $b = b(r)$ shapes the wormhole spatial geometry. The equation (3) shows two coordinate patches where each one covering the range of $r \in [r_0, +\infty)$. Thus, a coordinate patch is representing a universe. The two patches (universe) join at the throat [12] of the wormhole defined by r_0 where the geometry has a minimum radius $r = b(r) = r_0$. To avoid the presence of event horizon and the curvature singularity, $\Phi(r)$ must be everywhere finite [5]. The equation (4) also represent a flare-out condition of a wormhole that shows a vertical embedded surface when $r \to r_0$ at the throat where $dz/dr \to \infty$ and a horizontal embedded surface when $r \to \infty$ at a distance place far from the mouth of a wormhole where $dz/dr \to 0$.

As the solution to Einstein field equations, and after some calculations of Ricci and Einstein tensors, the solution result yield non-zero components of the stress-energy tensor ρ, τ and p denoted as energy density, radial tension and tangential pressure respectively:

$$\rho = \frac{b' c^4}{8\pi G r^2} \quad , \tag{5}$$

$$\tau = \frac{c^4}{8\pi G}\left[\frac{b}{r^3} - 2\left\{1 - \frac{b}{r}\right\}\frac{\Phi'}{r}\right] , \tag{6}$$

$$p = \frac{c^4}{8\pi G}\left\{\left(1 - \frac{b}{r}\right)\left[\Phi'' + \Phi'\left(\Phi' + \frac{1}{r}\right)\right] - \frac{1}{2r^2}(b'r - b)\left(\Phi' + \frac{1}{2}\right)\right\} . \tag{7}$$

These components in the stress energy tensor generate the space-time curvature of a wormhole.

3. Morris-Thorne wormhole with cosmological constant

It is known that the inflationary phase of the very-early universe and the currently accelerating and expanding universe demand the presence of cosmological constant; Λ, and from recent astronomical observation [13,14] it seems that we indeed live in a universe with Λ. Following the method of Morris-Thorne paper, Lemos et al. [15] analyzed the spherically symmetric and static traversable wormhole based on the space-time metric of Morris-Thorne (3) and the inclusion of Λ in the Einstein field equation where it can be rewritten as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} . \tag{8}$$

Finally, after solving (8) with calculation of the Einstein curvature tensor the following set of Einstein equations of state were obtained

$$\rho(r) = \frac{c^4}{8\pi G}\left(\frac{b'}{r^2} - \Lambda\right) , \tag{9}$$

$$\tau(r) = \frac{c^4}{8\pi G}\left[\frac{b}{r^3} - 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r} - \Lambda\right] \quad \text{and} \tag{10}$$

$$p(r) = \frac{c^4}{8\pi G}\left\{\left(1 - \frac{b}{r}\right)\left[\Phi'' + (\Phi')^2 - \frac{b'r - b}{2r^2(1 - b/r)}\Phi' - \frac{b'r - b}{2r^3(1 - b/r)} + \frac{\Phi'}{r}\right] + \Lambda\right\} . \tag{11}$$

These are the energy density, radial tension and tangential pressure respectively that generate the space-time curvature of a traversable wormhole in a universe with cosmological constant, Λ. The works by Lemos, Lobo and Oliveira [15] were indeed a generalization attempt from the works of Morris-Thorne.

4. Rotating wormhole

An extension of the Morris-Thorne traversable wormhole in term of its dynamical properties is to consider a stationary and axially symmetrical model as proposed by Teo [16] and Khatsymovsky [13]. The stationary and axially symmetric generalization would physically describe a rotation. Stationary space-time posses a time-like Killing vector field $\xi^a \equiv (\partial/\partial t)^a$ generating invariant time translation while the axially symmetric space-time posses a space-like Killing vector field $\psi^a \equiv (\partial/\partial\varphi)^a$ generating invariant rotations with respect to the angular coordinate.

From the work of Papapetrou [18] and Carter [19] the most general metric of a stationary and axially symmetrical model can be written as

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{ij}dx^i dx^j . \quad (12)$$

By this general space-time metric (12), Teo [16] introduced a general metric with a rotational identity

$$ds^2 = -N^2dt^2 + e^{\mu}dr^2 + r^2K^2[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2] \quad (13)$$

where the metric tensor $g_{\mu\nu}$ representing the four gravitational potentials that are r and θ dependent. The quantity ω is the angular velocity $\omega = d\varphi/dt$ of a particle that falls freely from infinity to the point (r,θ). From (13), specifically, the canonical metric for a stationary, axially-symmetric and traversable wormhole can be written as [16]

$$ds^2 = -N^2dt^2 + \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2K^2\left[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2\right] \quad (14)$$

where N, b, K and ω are only (r,θ) dependent and regular on the symmetry axis $\theta \in [0,\pi]$. This metric describes two identical asymptotically flat regions joined together at the throat where $r = b$. N is analogous to the red-shift function in equation (3) where it must always be finite and non-zero to ensure there is no event horizon or curvature singularity. The shape function b must satisfy $b \le r$ and at the throat $db/d\theta$ =0 which is independent of θ and obey flare-out condition $db/dr < 1$ to satisfy the equation $d^2r/dz^2 > 0$ derived from (4), K determined the proper radial distance $R = rK$, while ω governs the angular velocity of the wormhole.

Khatsymovsky proposed a rather similar space-time metric of (14) in a case of slow rotation where higher order of angular velocity is negligible i.e., $\omega^2 \rightarrow 0$, therefore, the space-time metric (14) reduced to a metric with small non-diagonal polar-angle-time components. Then, by taking $g_{tt} = e^{2\Phi}c^2$ as of Morris-Thorne, Khatsymovsky [17] proposed for the self-maintained stationary and slowly rotating wormhole space-time metric to be

$$ds^2 = -e^{2\Phi}c^2dt^2 + d\rho^2 + r^2(d\theta^2 + \sin^2\theta(d\varphi^2 + 2\omega d\varphi dt)). \qquad (15)$$

Khatsymovsky had considered the metric as the sum of spherically symmetric spacetime with small non-diagonal components, which describe small rotation ω. If $r \rightarrow \rho$ as $\rho \rightarrow \infty$, [17] thus, one can also specify as of Morris-Thorne that

$$d\rho^2 = \frac{dr^2}{1 - b/r}. \qquad (16)$$

5. Slowly rotating spherically symmetric wormhole with cosmological constant

A study of a wormhole solution is to be attempted based on the works of Morris-Thorne [5], Lemos et al. [15], Teo [16] and Khatsymovsky [17]. Even though Teo's metric tensors that represent the gravitational potentials $N, \mu,$ and his angular velocity ω are (r,θ) dependent, we would suggest, based on Morris-Thorne original metric that all potentials and angular velocity are independent of θ due to the conservation of red-shift function $\Phi(r)$,shape function $b(r)$ and $\omega(r)$ regardless of the variation of θ and therefore only dependent on r which is the spherically symmetric characteristic. Then by taking the potentials to be $N(r) = -e^{2\Phi(r)}$ and $\mu(r) = -\ln(1 - b(r)/r)$ as of the Morris-Thorne metric potentials, the general space-time metric with the characteristic of slow rotation reduces to

$$ds^2 = \left(-e^{2\Phi}c^2 + K^2r^2\omega^2\sin^2\theta\right)dt^2 + \frac{dr^2}{1 - \frac{b}{r}} + K^2r^2d\theta^2 + K^2r^2\sin^2\theta d\varphi^2 \qquad (17)$$

$$-K^2r^2\omega\sin^2\theta d\varphi dt$$

which is our proposed space-time metric for the slowly rotating spherically symmetric wormhole.

From (17), the metric tensors are

$$g_{00} = -e^{2\Phi}c^2 + K^2r^2\omega^2\sin^2\theta\ , \quad g_{11} = (1-b/r)^{-1}\ , \quad g_{22} = K^2r^2, \qquad (18)$$
$$g_{33} = K^2r^2\sin^2\theta\ , \qquad g_{03} = -K^2r^2\omega\sin^2\theta,$$

and the conjugate metric tensors are

$$g^{00} = -\frac{1}{e^{2\Phi}c^2}\ , \quad g^{11} = 1-b/r\ , \quad g^{22} = \frac{1}{r^2}\ ,$$
$$g^{33} = \frac{1}{K^2r^2\sin^2\theta} - \frac{\omega^2}{e^{2\Phi}c^2}\ , \quad g^{03} = -\frac{\omega}{e^{2\Phi}c^2}\ . \qquad (19)$$

Then we derive the affine connection that governs the acceleration of a freely falling particle in the gravitational field of the wormhole by the Christoffel symbol expression

$$\Gamma^{\alpha}_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right). \qquad (20)$$

Let us consider a slowly rotating wormhole that has a very small but significant magnitude of angular velocity ω which we assumed that $\omega^2 \cong 0$. Considering also the spherically symmetric wormhole of Morris-Thorne where $K = 1$, we obtained fourteen nonzero terms as the following

$$\Gamma^0_{01} = \Phi' - \frac{r^2\omega\omega'\sin^2\theta}{2e^{2\Phi}c^2},$$
$$\Gamma^0_{13} = \frac{r^2\omega'\sin^2\theta}{2e^{2\Phi}c^2},$$
$$\Gamma^1_{00} = \left(e^{2\Phi}c^2\Phi' - r^2\omega\omega'\sin^2\theta\right)\left(1-\frac{b}{r}\right),$$
$$\Gamma^1_{03} = \left(1-\frac{b}{r}\right)\left(\frac{r^2\omega'}{2} + r\omega\right)\sin^2\theta,$$
$$\Gamma^1_{11} = \frac{b'r-b}{2r(r-b)}, \qquad \Gamma^1_{22} = b-r,$$
$$\Gamma^1_{33} = (b-r)\sin^2\theta,$$

$$\Gamma^2_{03} = \omega \sin\theta\cos\theta\ , \qquad \Gamma^2_{12} = \frac{1}{r},$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta\ ,$$

$$\Gamma^3_{02} = -\omega\cot\theta\ , \qquad \Gamma^3_{13} = \frac{r^2\omega\omega'\sin^2\theta}{2e^{2\Phi}c^2} + \frac{1}{r} ,$$

$$\Gamma^3_{23} = \cot\theta\ ,$$

$$\Gamma^3_{01} = \omega\left(\Phi' - \frac{1}{r}\right) - \frac{\omega'}{2}\ . \tag{21}$$

The others are null affine connections whose metric tensors are independent to the dynamic parameters of the test particles.

From all these we can construct the Ricci tensor in which represent the difference of acceleration between test particles due to the space-time curvature given by the expression

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\nu}\ . \tag{22}$$

The Ricci tensor is symmetrical where $R_{\mu\nu} = R_{\nu\mu}$, so it has ten different components, they are; R_{00}, R_{01}, R_{02}, R_{03}, R_{11}, R_{12}, R_{13}, R_{22}, R_{23} and R_{33}. By all the affine connection fourteen terms, we have calculated all the ten components of the Ricci tensor. Then contracting the conjugate metric tensor we obtained the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ where,

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + 2g^{03}R_{03}\ . \tag{23}$$

From the Ricci tensors and Ricci scalar we can construct the Einstein tensor which is known as the curvature tensor $G_{\mu\nu}$ that represent the space-time curvature which also representing the gravitational field of the wormhole. The Einstein tensor relationship with Ricci tensor and Ricci scalar are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\ . \tag{24}$$

Einstein field equations show the proportional relationship between the space-time curvature $G_{\mu\nu}$ and the energy momentum tensor $T_{\mu\nu}$ which represent the gravitational field and the energy-matter respectively by $G_{\mu\nu} = 8\pi Gc^{-4}T_{\mu\nu}$.

With the inclusion of the cosmological constant the relationship then become $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$. From the equation of perfect fluid the energy momentum tensor is given by $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$. Combining the Einstein field equation with the perfect fluid equation we will have

$$(\rho + p)u_\mu u_\nu + p g_{\mu\nu} = \frac{c^4}{8\pi G}\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right) \quad (25)$$

where $u_\mu = \frac{dx_\mu}{d\tau} = \frac{dx_\mu}{dt}\frac{dt}{d\tau}$ and $\frac{dt}{d\tau} = \gamma$ where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. The velocity of test particle is regarded to be very much smaller than the light speed $v << c$, therefore $\gamma = 1$. The test particle's reference frame follows the geodesic of the wormhole and thus does not necessarily by itself rotate therefore we can consider that $d\varphi/d\tau = 0$. With $x_\mu = (ct, r, \theta, \varphi)$, thus $u_\mu = (c, 0, 0, 0)$ or we can write $u_0 = c$, $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$.

From (25) we derive the Einstein equations of state that are for the energy density, radial tension and the tangential pressure of the wormhole by analyzing the energy momentum tensors T_{00}, T_{11}, and T_{22} which are related to the calculation of the equations. Finally, we obtained the expression of the energy density ρ, the radial tension τ and the tangential pressure p as the following:

$$\rho = \frac{c^4}{8\pi G}\left(\frac{b'}{r^2} - \Lambda - \left(1 - \frac{b}{r}\right)\left\{r^2\left(\omega\omega'' - \omega\omega'\left[\frac{b'r - b}{2r(r-b)} + \Phi'\right]\right) + 3r\omega\omega' + \frac{r^2\omega'^2}{4}\right\}\frac{\sin^2\theta}{c^2}\right), \quad (26)$$

$$\tau = \frac{c^4}{8\pi G}\left(\frac{b}{r^3} - 2\frac{\Phi'}{r}\left(1 - \frac{b}{r}\right) - \Lambda - \frac{r^2}{4c^2}\left(1 - \frac{b}{r}\right)\left(\omega'^2 + \frac{4\omega\omega'}{r}\right)\sin^2\theta\right), \quad (27)$$

$$p = \frac{c^4}{8\pi G}\left(\left(1 - \frac{b}{r}\right)\left(\Phi'' + \Phi'^2 - \frac{(b'r - b)}{2r^2\left(1 - \frac{b}{r}\right)}\Phi' + \frac{b'r - b}{2r^3\left(1 - \frac{b}{r}\right)} + \frac{\Phi'}{r}\right) + \Lambda\right.$$

$$\left. - \frac{r^2}{4c^2}\left(1 - \frac{b}{r}\right)\left(\omega'^2 - \frac{4\omega\omega'}{r}\right)\sin^2\theta\right). \quad (28)$$

The above equations consist of the non-rotating Morris-Thorne terms and the rotating terms. Thus, it can be shown that if the rotation comes to a halt or if there is no rotation at all, then all the expressions above will reduce to that of the non-rotating Morris-Thorne wormhole with cosmological constant as proposed by Lemos et al. These equations represent the governing physical state of the wormhole and it is useful for the detail study of regions in the wormhole namely, the interior, the junction condition (mouth) and the exterior.

We can now derive the spacetime metric or the line element for the exterior solution of the wormhole which we define the line element as what an observer would view the wormhole externally. Comparing (14) with the Schwarzchild line element (1) and we may write in general the line element as

$$ds^2 = -e^{\nu}c^2dt^2 + e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta[d\varphi - \omega dt]^2\right) \qquad (29)$$

Therefore let $\nu = 2\Phi$ and $e^{-\lambda} = 1 - \frac{b}{r}$. Since the rotating effect of the wormhole reduces significantly at the external region ($\omega \to 0$ as $r \to \infty$), ω' must be significant. Hence, to this at radii near the mouth we have $\omega\omega' \cong 0$ and $\omega\omega'' \cong 0$ which defines a slim wormhole. We approximate (26) the density close to the mouth to be

$$\rho \cong \frac{c^4}{8\pi G}\left(\frac{b'}{r^2} - \Lambda - \left(1 - \frac{b}{r}\right)\frac{r^2\omega'^2}{4c^2}\sin^2\theta\right) \qquad (30)$$

Taking $\rho = m/V$ that is the mass over the volume of the wormhole and considering the spherically symmetric nature of the wormhole we can consider $V = 4\pi r^3/3$. From $m = 4\pi\rho r^3/3$ we can write $m = 4\pi\rho\int r^2 dr$ and therefore $d(2m) = 8\pi\rho r^2 dr$. Thus

$$\frac{8\pi G}{c^4}\rho = \frac{G}{c^4 r^2}\frac{d(2m)}{dr} . \qquad (31)$$

As $e^{-\lambda} = 1 - \frac{b}{r}$, therefore

$$b' = 1 - r\frac{d\left(e^{-\lambda}\right)}{dr} - e^{-\lambda} . \qquad (32)$$

By combining (30), (31) and (32) we have

$$\frac{G}{c^4 r^2}\frac{d(2m)}{dr} = \frac{1}{r^2}\left(1 - r\frac{d\left(e^{-\lambda}\right)}{dr} - e^{-\lambda}\right) - \Lambda - \left(1 - \frac{b}{r}\right)\frac{r^2\omega'^2}{4c^2}\sin^2\theta \qquad (33)$$

In term of $e^{-\lambda}$ we have

$$e^{-\lambda}=1-\frac{2mG}{rc^4}-\frac{\Lambda r^2}{3}-\frac{\sin^2\theta}{4c^2}\int\left(1-\frac{b}{r}\right)r^3\omega'^2\,dr \quad (34)$$

Extending to the exterior region and since the spacetime is almost flat $b_{ext}\cong 0$, we have

$$e^{-\lambda}=1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\sin^2\theta}{4c^2}\int r^3\omega'^2_{ext}\,dr\,,$$

$$=1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\sin^2\theta}{4c^2}\left(\frac{\omega'^2_{ext}r^4}{4}-\frac{\omega''_{ext}}{2}\int r^4 d\omega\right) \quad (35)$$

Considering that ω''_{ext} is finite in this exterior region albeit a small value as $\omega_{ext}\to 0$, and $\Delta\omega_{ext}\to d\omega_{ext}\cong 0$ as $r\to\infty$ (35) reduces to

$$e^{-\lambda}=1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}$$

or

$$e^{\lambda}=\left(1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}\right)^{-1}. \quad (36)$$

With $\omega_{ext}\cong 0$, (29) can be written as

$$ds^2=-e^{\nu}c^2dt^2+\frac{dr^2}{1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}}+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right). \quad (37)$$

We can now solve for the expression of e^{ν} by using the contracted Bianchi identities for analyzing all the non zero Einstein tensors. Recall that the contracted Bianchi identities are $\nabla_\gamma G^\gamma_\nu=0$ where the mixed Einstein tensor is defined as $G^\gamma_\nu=g^{\gamma\mu}G_{\mu\nu}$. Applying the identities, then

$$\nabla_0 G^0_\nu+\nabla_1 G^1_\nu+\nabla_2 G^2_\nu+\nabla_3 G^3_\nu=0. \quad (38)$$

With the fact that only G^0_0, G^1_1, G^2_2, G^3_3, G^0_3 and G^3_0 are the non-zero mixed Einstein tensors we have for $\nu=0$,

$$\nabla_0 G^0_0+\nabla_3 G^3_0=0. \quad (39)$$

Since,

$$\nabla_3 G^3_0=\frac{dG^3_0}{d\varphi}+G^\gamma_0\Gamma^3_{\gamma 3}-G^3_\gamma\Gamma^\gamma_{03}$$

$$=\frac{dG_0^3}{d\varphi}+G_0^0\Gamma_{03}^3+G_0^1\Gamma_{13}^3+G_0^2\Gamma_{23}^3+G_0^3\Gamma_{33}^3-G_0^3\Gamma_{03}^0-G_1^3\Gamma_{03}^1-G_2^3\Gamma_{03}^2-G_3^3\Gamma_{03}^3=0, \quad (40)$$

therefore

$$\nabla_0 G_0^0=0. \quad (41)$$

For $\nu=1$,

$$\nabla_1 G_1^1=0. \quad (42)$$

For $\nu=2$,

$$\nabla_2 G_2^2=0. \quad (43)$$

For $\nu=3$,

$$\nabla_0 G_3^0+\nabla_3 G_3^3=0. \quad (44)$$

Since

$$\nabla_0 G_3^0=\frac{dG_3^0}{dt}+G_3^\gamma\Gamma_{\gamma 0}^0-G_\gamma^0\Gamma_{30}^\gamma$$

$$=\frac{dG_3^0}{dt}+G_3^0\Gamma_{00}^0+G_3^1\Gamma_{10}^0+G_3^2\Gamma_{20}^0+G_3^3\Gamma_{30}^0-G_0^0\Gamma_{30}^0-G_1^0\Gamma_{30}^1-G_2^0\Gamma_{30}^2-G_3^0\Gamma_{30}^3=0, \quad (45)$$

therefore

$$\nabla_3 G_3^3=0. \quad (46)$$

Thus from (41) to (46) it can also be suggested that all the previously nonzero mixed Einstein tensors vanish that is, G_0^0, G_1^1, G_2^2, G_3^3, G_3^0 and G_0^3 are all now equal to zero.

We can derive the relationship between e^λ and e^ν for the exterior solution by analyzing G_0^0 and G_1^1.

For $G_0^0=0$, where $G_0^0=g^{00}G_{00}$ and $b=b_{ext}$ in the external region, it can be shown that

$$-\frac{b'_{ext}}{r^2}+\left(1-\frac{b_{ext}}{r}\right)\frac{r^2\omega'^2_{ext}}{4c^2}\sin^2\theta=0. \quad (47)$$

By (33) and taking $e^{-\lambda}=1-\frac{b_{ext}}{r}$ we have

$$\frac{1}{r^2}-\frac{e^{-\lambda}}{r^2}+\frac{e^{-\lambda}\lambda'}{r}-\frac{e^{-\lambda}r^2\omega'^2_{ext}}{4c^2}\sin^2\theta=0. \quad (48)$$

For $G_1^1=0$, where $G_1^1=g^{11}G_{11}$ it can be shown that

$$-\frac{b}{r^3}+\frac{2\Phi'}{r}\left(1-\frac{b_{ext}}{r}\right)+\left(1-\frac{b_{ext}}{r}\right)\frac{r^2\omega'^2_{ext}}{4c^2}\sin^2\theta=0. \quad (49)$$

Again, by (33) taking $e^{-\lambda}=1-\frac{b_{ext}}{r}$ and $\nu'=2\Phi'$ we have

$$-\frac{1}{r^2}+\frac{e^{-\lambda}}{r^2}+\frac{e^{-\lambda}\nu'}{r}+\frac{e^{-\lambda}r^2\omega'^2_{ext}}{4c^2}\sin^2\theta=0. \qquad (50)$$

Adding (48) and (50) we get

$$\frac{e^{-\lambda}\lambda'}{r}+\frac{e^{-\lambda}\nu'}{r}=0. \qquad (51)$$

Thus,

$$\lambda'+\nu'=0 \qquad (52)$$

and integrating gives

$$\lambda+\nu=h(t) \qquad (53)$$

or by exponential function we can write

$$e^{\lambda}e^{\nu}=e^{h(t)}. \qquad (54)$$

To eliminate $h(t)$ we transform to a new time coordinate t', i.e. $t\rightarrow t'$ where

$$t'=\int_k^t e^{\frac{1}{2}h(u)}du \qquad (55)$$

and k is an arbitrary constant.

As the result of eliminating $h(t)$ we get $h(t)=0$ and thus $e^{h(t)}=1$, therefore

$$e^{\nu}=e^{-\lambda}.$$

Finally we get

$$e^{\nu}=1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}. \qquad (56)$$

Thus, (37) can be written as

$$ds^2=-\left(1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}\right)c^2dt^2+\frac{dr^2}{1-\frac{2mG}{rc^4}-\frac{\Lambda_{ext}r^2}{3}-\frac{\omega'^2_{ext}r^4\sin^2\theta}{16c^2}}$$
$$+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right). \qquad (57)$$

This is a unique external solution for a slowly rotating spherically symmetric wormhole with a cosmological constant.

6. Conclusion

The space-time metric for the spherically symmetric wormhole derived from the combination of Morris and Thorne, Khatsymovsky, and Teo is considered stationary and thus, is time independent. The spacetime metric has a simplistic character of Morris-Thorne wormhole and rotating character of slowly rotating wormhole space-time metric of Khatsymovsky, and at the foundation of it, are the more generalized character of rotating wormhole space-time metrics by Teo. In the framework of classical general relativity we have introduced the slowly rotating spherically symmetric wormhole with cosmological constant. It is indeed the generalization of the pioneering works from Einstein and Rosen until the recent works i.e., of Lemos et al. and Teo. From the equations of perfect fluid representing the energy momentum tensors an important stage in these derivations is executed with the inclusion of the cosmological constant. Relating the energy momentum tensors with the Einstein tensors, we have derived Einstein equations of state. These equations can be separated nicely between the non-rotating terms and the rotating term and thus show that if no rotational effect occurs the character of the equations will reduce to that of Morris-Thorne wormhole with cosmological constant as proposed by Lemos et al.. Finally from the energy density equation and comparing with the the Schwarzchild line element we derived the unique external solution for the slowly rotating spherically symmetric wormhole with a cosmological constant. The equation (57) above shows that, at the external region of vacuum with no mass and cosmological constant, the spacetime metric reduce to that of Minkowski flat spacetime. With the presence of matter but without cosmological constant then the spacetime metric reduces to that of Schwarzchild spherically symmetric spacetime. If there is a presence of cosmological constant, but with no rotation, then the spacetime metric reduce similarly to that of Morris-Thorne wormhole with Λ.

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