MAXIMAL TEMPORAL AMPLITUDE (MTA) OF TRI-CHROMATIC SIGNALS

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Abstract

Previous numerical and experimental results about surface waves which are originally bi-chromatic signal on the wave maker, show phenomenon about peaking and splitting during their propagation in the wave tank. To investigate this phenomenon, especially in peaking, a quantity called Maximal Temporal Amplitude (MTA) is introduced. This quantity can be used to predict the position of the occurrence of maximal peaking. In this paper, we present the MTA form of tri-chromatic signal, especially to predict the position of maximal peaking of the signal. Here, the tri-chromatic signal is a superposition of three mono-chromatic signals with different of amplitudes, frequencies and initial phases. We use KdV equation with exact dispersion relation (linear) as a model. We get some approximated solution by expanding the equation in the power series of amplitude and wave number of tri-chromatic signal up to third order.

Keywords : Phase, tri-chromatic signal, Maximal Temporal Amplitude

1. Introduction

This study is motivated by hydrodynamic laboratories needs to generate "extreme waves" deterministically, at some position in the wave tank. The result will be used in ship and floating body structures testing that operate against the extreme waves. These waves are extraordinarily large water waves whose heights exceed by a factor the significant wave height of a measured wave train [5]. Also, these waves are very difficult to predict and very dangerous for some ships and floating body structures (see [6,7,10]). For that, we need to generate the waves with this characteristic. On the other hand, there is a problem about what kind of signal we have to generate at the wave maker such that, at some position the signal has some extreme deformations.

From the previous known results (as the effects of medium non-linearity, such as water) about surface waves, which are bi-chromatic waves, at the wave maker there is a phenomenon about peaking and splitting during their propagation in the wave tank (see

[9,11,13,14]). To investigate this phenomenon, especially in peaking, we introduce a quantity, called Maximal Temporal Amplitude (MTA). This quantity is used to measure the signal maximal amplitude at every position. We also know, from MTA, the position such that a maximal peaking occurs [2,9].

Some exact solution of Non Linear Schrödinger (NLS) equation, called Soliton on Finite Background (SFB), are extension of Benjamin-Feir wave models, that is, some mono-chromatic wave that modulated by long wave with small frequency modulations (included in Benjamin-Feir instability intervals) [1]. This elevated SFB MTA value at some position could reach three times their background MTA value if their modulated frequency is very small [8]. This property is very interesting for extreme wave generation.

From the above paragraph, we count MTA by approximated solution of KdV equation up to third order, using three dominant frequencies of SFB signal spectrum at the wave maker. But, as a result we have a position inconsistency where the generated tri-chromatic signal having some extreme deformations. May be, this inconsistency is caused by initial phase effects from all of the three dominant mono-chromatic SFB signal as a boundary condition of approximated solution of KdV equation up to third order at the wave maker.

In this paper, we will present the form of MTA of tri-chromatic signals. The signal as a superposition of three mono-chromatic signals with different of amplitudes, frequencies, and initial phases. We use KdV equation with exact dispersion relation (linear) as a model. We get some approximated solution by direct expansions in power series form of amplitude and wave number of tri-chromatic signal up to third order.

The organization of this paper as follow. In the next section, we shall briefly the model equation of surface gravity waves. Then direct third order approximated solution of KdV equation are given in section 3. In section 4, we discuss Maximal Temporal Amplitude and its maximum positions. Graphical results we present in section 5. Finally, we end this paper with conclusion.

2. Model of surface gravity waves

The elevation of rather long and small surface gravity waves were governed by Korteweg de Vries (KdV) equation. In normalized variable, the KdV equation has the form

$$\partial_t \eta + i\Omega(-i\partial_x)\eta = -\frac{3}{4}\partial_x \eta^2 \qquad (1)$$

with $\eta(x,t)$ is wave elevation. Here, $\Omega(k) = k\sqrt{tanh\, k / k}$ is the exact dispersion relation between frequency ω and wave number k [12]. The corresponding laboratory variables η_{lab}, x_{lab}, t_{lab} and normalized variables should be found from the transformations $\eta_{lab} = h\eta$, $x_{lab} = hx$ and $t_{lab} = t\sqrt{h/g}$, where h is the uniform depth water and g is gravity accelerations. The spatial variable x denotes the horizontal direction and t is temporal variable. The corresponding transformed wave parameters such as wave length, wave number and angular frequency, are given by $\lambda_{lab} = h\lambda$, $k_{lab} = k/h$, $\omega_{lab} = \omega\sqrt{g/h}$. These transformations normalized the depth of the layer and the propagation speed of infinitesimal long waves.

3. Third order asymptotic expansions

In this paper, we will solve (1) using a direct expansions up to third order in the power series form of the elevation signal amplitude. Here, we write

$$\eta = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \varepsilon^3 \eta^{(3)}, \tag{2}$$

ε is a positive small number, up to first order, representing the magnitude of wave amplitude. The terms $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$ are describe the linear, second order non linear and the third order side band non linear solutions, respectively. It is known, using this method gives a result to resonance in the third order [4]. To prevent the resonant term, we modify this expansions using Linstead-Poincare technique [15]

$$k = k^{(0)} + \varepsilon k^{(1)} + \varepsilon^2 k^{(2)} \tag{3}$$

where $k^{(0)} = \Omega^{-1}(\omega)$. We take a tri-chromatic wave as a solution of (1) i.e. a solution of $\eta^{(1)}$, that involves three frequencies

$$\eta^{(1)}(x,t) = \sum_{p=1}^{3} a_i e^{i\theta_p} + c.c., \tag{4}$$

where $\theta_p = k_p x - \omega_p t + \varphi_p$, with φ_p is the initial phases of mono-chromatic components and c.c. denotes the complex conjugate of the previous terms. The following procedure has been described in [3,4] without taking the initial phases of monochromatic components, i.e., $\varphi_p = 0$. Substituting (2) and (3) into (1) we obtain $\eta^{(1)}$ as the solution of the linearized equation as in (4). In the second order (ε^2), $k_p^{(1)} = 0$, $p=1,2,3$ and

$$\eta^{(2)} = \frac{3}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} a_p a_q (s_+ e^{i(\theta_p + \theta_q)} + s_- e^{i(\theta_p - \theta_q)}) + c.c., \tag{5}$$

where $s_\pm = \dfrac{1}{2} \dfrac{k_p^{(0)} \pm k_q^{(0)}}{\omega_p \pm \omega_q - \Omega(k_p^{(0)} \pm k_q^{(0)})}$.

In order to distinguish the free waves that will be introduced later, we call the second order solution in (5) the second order bound wave; this solution contains non linear terms as the results of mode generation. The resonant term in third order lead to the definition of non-linear dispersion relation $k_p^{nl} = k_p^{(0)} + k_p^{(2)}$ with

$$k_p^{(2)} = -\frac{9k_p^{(0)}}{4V_g(k_p^{(0)})} \sum_{q=1}^{3} a_q^2 (s_+ + s_-) \; ; p = 1,2,3 . \tag{6}$$

On the other hand, the non resonant term in the third order contains frequency components as well as components close to the resonant terms, called third order side band. The third order side band can be expressed as

$$\begin{aligned} \eta_{sb}^{(3)} = & \frac{9}{4} \sum_{p=q}^{3} \sum_{q \neq r}^{3} \sum_{r=1}^{3} a_p a_q a_r L_{pq,r} (s_p + s_{q,r}) e^{i(\theta_p + \theta_q - \theta_r)} + \\ & \frac{9}{8} \sum_{p \neq q, p \neq r}^{3} \sum_{q \neq r}^{3} \sum_{r=1}^{3} a_p a_q a_r L_{pq,r} (s_{p,r} + s_{pq} + s_{q,r}) e^{i(\theta_p + \theta_q - \theta_r)} + c.c., \end{aligned} \tag{7}$$

where

$$s_p = \frac{k_p^{(0)}}{2\omega_p - \Omega(2k_p^{(0)})},\ s_{pq} = \frac{1}{2}\frac{k_p^{(0)} + k_q^{(0)}}{\omega_p + \omega_q - \Omega(k_p^{(0)} + k_q^{(0)})}$$

$$s_{p,r} = \frac{k_p^{(0)} - k_r^{(0)}}{\omega_p - \omega_r - \Omega(k_p^{(0)} - k_r^{(0)})},\ L_{pq,r} = \frac{k_p^{(0)} + k_q^{(0)} - k_r^{(0)}}{\omega_p + \omega_q - \omega_r - \Omega(k_p^{(0)} + k_q^{(0)} - k_r^{(0)})}$$

and $V_g(k_p^{(0)}) = \Omega'(k_p^{(0)}), p = 1,2,3$.

In this paper, we are interest in a solution that has an initial signal at certain position, $x=0$, given by $\eta(0,t) = \sum_{p=1}^{3} a_i e^{i\theta_p} + c.c.$, Then, we have a boundary value problem of (1) with

$$\eta(0,t) = \sum_{p=1}^{3} a_i e^{i\theta_p} + c.c. \tag{8}$$

To satisfy initial signal, the contribution of the second order and third order side band terms at certain position have to be compensated by harmonic modes (free waves). The second order free waves can be written as

$$\eta_{free}^{(2)} = \frac{3}{2}\sum_{p=1}^{3}\sum_{q=1}^{3} a_p a_q (s_+ e^{i\vartheta(\omega_p+\omega_q)} + s_- e^{i\vartheta(\omega_p-\omega_q)}) + c.c. \tag{9}$$

This is wave with the same frequencies in the second order bound wave but consisting of harmonic modes that satisfy the linear dispersion relation. The third order side band free waves consists similarly of monochromatic waves and is of the form

$$\begin{aligned}\eta_{sb,free}^{(3)} = &\frac{9}{4}\sum_{p=q}^{3}\sum_{q\neq r}^{3}\sum_{r=1}^{3} a_p a_q a_r L_{pq,r}(s_p + s_{q,r}) e^{i\vartheta(\omega_p+\omega_q-\omega_r)} + \\ &\frac{9}{8}\sum_{p\neq q, p\neq r}^{3}\sum_{q\neq r}^{3}\sum_{r=1}^{3} a_p a_q a_r L_{pq,r}(s_{p,r} + s_{pq} + s_{q,r}) e^{i\vartheta(\omega_p+\omega_q-\omega_r)} + c.c.,\end{aligned} \tag{10}$$

with $\vartheta(\omega_p) = \Omega^{-1}(\omega_p)x - \omega_p t + \varphi_p;\ p = 1,2,3$. Taken together, the third order solution of (1) which is satisfying (8) is

$$\eta = \eta^{(1)} + \eta^{(2)} - \eta_{fw}^{(2)} + \eta_{sb}^{(3)} - \eta_{sb,free}^{(3)}.$$

4. Maximal Temporal Amplitude (MTA) and its extreme position

Using the third order approximations derived above, we investigate the phase shift effects on Maximal Temporal Amplitude (MTA) of tri-chromatic signals. According to previous section Maximal Temporal Amplitude (MTA) is defined by

$$m(x) = \max_t \eta(x,t).$$

In this paper, as a special case we choice $a_1=a_3=\delta a_2$, $\omega_1=\omega_0+\nu$, $\omega_3=\omega_0-\nu$ with $a_2=a_0$ and $\omega_2=\omega_0$. So that, (8) can be written as

$$\eta(0,t) = a_0 e^{i(-\omega_0 t+\varphi_0)} + \delta a_0 e^{i(-\omega_1 t+\varphi_1)} + \delta a_0 e^{i(-\omega_3 t+\varphi_3)} + c.c. \quad (11)$$

where φ_0, φ_1 and φ_3 are the initial phases of monochromatic components. For ease of presentation, we rewrite the phases φ_0, φ_1 and φ_3 as $\varphi_1=\varphi_0+\alpha+\Delta\alpha$ *and* $\varphi_3=\varphi_0+\alpha-\Delta\alpha$, with arbitrary values of φ_0, α and $\Delta\alpha$.

$$\eta(0,t) = a_0(1+2\delta e^{i\alpha}\cos(-\nu t+\Delta\alpha))e^{i(-\omega_0 t+\varphi_0)} + c.c. \quad (12)$$

The expression of $\eta^{(1)}$ in (4) becomes

$$\eta^{(1)}(x,t) = a_0[1+2\delta e^{i((\bar{k}^{nl}-k_0^{nl})x+\alpha)}\cos(\kappa^{nl}x-\nu t+\Delta\alpha)]e^{i(k_0^{nl}x-\omega_0 t+\varphi_0)} + c.c, \quad (13)$$

where $\bar{k}^{nl} = (k_1^{nl}+k_3^{nl})/2$, $\kappa^{nl} = (k_1^{nl}-k_3^{nl})/2$ and k_0^{nl} the non linear wave number corresponding to ω_0. From (13), we observe that the superposition of the three monochromatic waves lead to spatial envelope that is modulated with wave length $\lambda_1 = 2\pi/\left|\bar{k}^{nl}-k_0^{nl}\right|$.

As shown in [2,9], the second order terms do not affect the extreme position; hence this position can be approximated by considering the first and the third order side bands only. The most important third order side bands that determine the position are terms in (7) that have wave numbers close to that of the first order term. These important terms as a part of $\eta_{sb}^{(3)}(x,t)$ can be written in the form

$$\begin{aligned}\eta^{*(3)}_{sb}(x,t) = {} & a_0^3\rho e^{i((k_3^{nl}+k_1^{nl}-k_0^{nl})x-\omega_0 t+2\alpha)} + a_0^3 B_{-3}\, e^{i((2k_0^{nl}-k_1^{nl})x-(\omega_0-\nu)t-\alpha+\Delta\alpha)} + \\ & a_0^3 B_{-3}\, e^{i((2k_0^{nl}-k_3^{nl})x-(\omega_0+\nu)t-\alpha-\Delta\alpha)} + c.c,\end{aligned} \quad (14)$$

and the corresponding terms of $\eta_{sb,free}^{(3)}(x,t)$ are

$$\begin{aligned}\eta^{*(3)}_{sb,free}(x,t) = {} & a_0^3\rho e^{i(\Omega^{-1}(\omega_0)x-\omega_0 t+2\alpha)} + a_0^3 B_{-3}\, e^{i(\Omega^{-1}(\omega_0-\nu)x-(\omega_0-\nu)t-\alpha+\Delta\alpha)} + \\ & a_0^3 B_{-3}\, e^{i(\Omega^{-1}(\omega_0+\nu)x-(\omega_0+\nu)t-\alpha-\Delta\alpha)} + c.c,\end{aligned} \quad (15)$$

here $\rho = \delta^2\mu^2(s_{1,0}+s_{3,0}+s_{13})L_{13,0}$, $B_{-3} = \delta\mu^2(s_{01}+s_0)L_{00,1}$, $B_{+3} = \delta\mu^2(s_{03}+s_0)L_{00,3}$. Since $\delta<<1$ it follows that $\rho<<B_{\pm 3}$ and so the terms $a_0^3\rho e^{i((k_3^{nl}+k_1^{nl}-k_0^{nl})x-\omega_0 t+2\alpha)}$ in (14) and $a_0^3\rho e^{i(\Omega^{-1}(\omega_0)x-\omega_0 t+2\alpha)}$ in (15) can be neglected. The remaining terms in (14) and (15) become

$$\eta^{*(3)}_{sb} \approx a_0^3 e^{-i[(\bar{k}^{nl}-k_0^{nl})x+\alpha]} B_{bw}(x,t) e^{i[k_0^{nl}x-\omega_0 t+\varphi_0]} + c.c. \quad (16)$$

and

$$\eta^{*(3)}_{sb,free} \approx a_0^3 e^{-i[-\frac{1}{2}K''(\omega_0)\nu^2 x+\alpha]} B_{fw}(x,t) e^{i[K(\omega_0)x-\omega_0 t+\varphi_0]} + c.c. \quad (17)$$

where

$$B_{bw}(x,t) = B_{-3}e^{i(\kappa^{nl}x-\nu t+\Delta\alpha)} + B_{+3}e^{-i(\kappa^{nl}x-\nu t+\Delta\alpha)},$$

$$B_{fw}(x,t) = B_{-3}e^{i(K'(\omega_0)\nu x-\nu t+\Delta\alpha)} + B_{+3}e^{-i(K'(\omega_0)\nu x-\nu t+\Delta\alpha)},$$

$K(\omega_0) = \Omega^{-1}(\omega_0)$, and $K(\omega_0\pm\nu) \approx K(\omega_0) \pm K'(\omega_0)\nu + \frac{1}{2}K''(\omega_0)\nu^2$.

It is interesting to see that the third order bound and free waves have spatial envelopes that do not depend on time. The first order term has an envelope with the same length as the third order bound wave $\eta_{sb}^{*(3)}(x,t)$. Thus superposition of the first order term with the third order bound and free waves lead to a spatial envelope with a short modulation length $\lambda_s = 4\pi / \left\{ \left| -\frac{1}{2}K''(\omega_0)\nu^2 \right| + \left| \bar{k}^{nl} - k_0^{nl} \right| \right\}$ and a longer modulation length $\lambda_l = 4\pi / \left\{ \left| -\frac{1}{2}K''(\omega_0)\nu^2 \right| - \left| \bar{k}^{nl} - k_0^{nl} \right| \right\}$. These values of λ_s and λ_l are obtained by only considering the superposition of the third order bound waves and free waves where λ_s shows how MTA oscillates with λ_l as the overall wave length of the MTA. Furthermore, since the spatial envelopes of the third order bound waves and free waves contain the phase of α, the location of the first maximal position of the MTA can be expressed in the form

$$\hat{x}_{max} = \frac{\pi}{\left| -\frac{1}{2}K''(\omega_0)\nu^2 \right| + \left| \bar{k}^{nl} - k_0^{nl} \right|} - \frac{2\alpha}{-\frac{1}{2}K''(\omega_0)\nu^2 + \bar{k}^{nl} - k_0^{nl}} \tag{18}$$

5. Some Graphical Results

In the following, we will present some graphical results based on the explicit third order solution derived at the previous section. Particularly, we will show MTA as a function of spatial variables, x, and the maximum value of that function can reach. As an example, we take μ=3/2, a_0=0.08m, δ=0.1, ω_0=3.145/s and ν=0.155/s for some the initial phases of three mono-chromatic components. In Figure 1 we plot the MTA as a function of spatial variables x for $\varphi_0 = 0, \alpha = 0$ and $\Delta\alpha = 0$. In fact, in each sub-figure, there are two parts: the upper one for MTA of all orders and the lower one when the second order terms are suppressed. According to the figures, the second order terms contribute to high oscillation for $\eta(x,t)$. On the other hand, it is clear from that figures, the position of MTA maximum value can be determined through the first order and third order side band interactions. For the case above the first position of is $x_{max} = 180$ m while the approximate $\hat{x}_{max} = 185$ m. We will now investigate how the initial phases of mono-chromatic components influence the value x_{max} and $\hat{x}_{max}$. In Table 1, we show the value x_{max} and the predicted value $\hat{x}_{max}$ for the same case above but different value of α. In Figure 2, we show $m_{-2}(x)$ for three different values of α, the upper figure for α=0, the middle figure for α=π/4 and the lower one for α=π/6. It is clearly seen that there is a significant difference of the location where the maximum value is obtained.

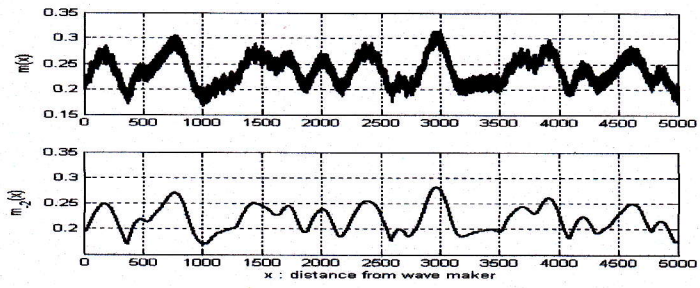

Figure 1. *The Maximal Temporal Amplitude calculated in the upper figure with all orders, $m(x)$, and the lower figure without the second order contributions, $m_{-2}(x)$*

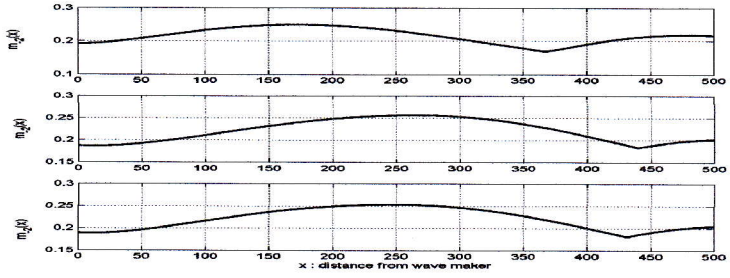

Figure 2. *Maximal Temporal Amplitude for three different values of α. (Upper) $\alpha=0$, (middle) $\alpha=\pi/4$ and (lower) $\alpha=\pi/6$*

Table 1. The first maximum x_{max} of $m_{-2}(x)$ and the approximated value $\hat{x}_{max}$ in (18) for various values of α.

Case	x_{max}	$\hat{x}_{max}$
α=π/3	302.0m	308.40m
α=π/4	280.0m	277.60m
α=π/5	261.5m	259.10m
α=π/6	246.5m	246.75m

6. Conclusion

This paper aims to contribute to the extreme wave generation in typical wave tanks of hydrodynamic laboratories. We have considered a class of wave groups, namely Tri-chromatics as a superposition of three mono-chromatic with different of amplitudes, frequencies and the initial phases, generated running on still water. Propagation of prescribed signal at the wave maker for the class is investigated to see if maximal peaking occurs and what position in the wave tank. We have introduced a quantity called Maximal Temporal Amplitude (MTA) using a KdV type of equation based on its third order

approximated solution. According the quantity can be known the position of the occurrence of maximal peaking. We showed the initial phases of mono-chromatic components influence the location of the maximum MTA considerably.

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