THE DOWNSIDE RISK OPTIMAL PORTFOLIO SELECTION PROBLEM

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Abstract

One of the basic problems of applied finance is the optimal selection of stocks, with the aim of maximizing future returns and minimizing the risk using a specified risk aversion factor. Variance is used as the risk measure in classical Markowitz model, thus resulting in a quadratic programming. As an alternative, mean absolute deviation was proposed as a risk measure to replace the original risk measure, variance. This problem is a straight-forward extension of the classic Markowitz mean-variance approach and the optimal portfolio problem can be formulated as a linear programming problem. Taking the downside risk as the risk leads to different optimal portfolio. The effect of using only downside risk on optimal portfolio is analyzed in this paper by taking the **mean absolute negative deviation** as the risk measure. This method is applied to the optimal selection of stocks listed in Bursa Malaysia and the return of the optimal portfolio is compared to the classical Markowitz model and mean absolute deviation model. The result show that the optimal portfolios using downside risk measure outperforms the other two models.

Keywords:-Portfolio optimization, Linear Programming, Downside risk.

1. Introduction

Portfolio optimization problem has been one of the important research fields in modern finance. Generally, investors always prefer to have the return on their portfolio as large as possible. At the same time, he also wants to make the risk as small as possible. However, some investors pursue a high return even though it is accompanied with a higher risk.

The basic theory of portfolio optimization was presented by Markowitz in his pioneering article [3]. By employing the standard deviation and expected value of the stocks as the representation of return, Markowitz introduced the famous mean-variance model, which has been regarded as a quadratic programming problem. There has been a tremendous amount of researches on improving this basic model both computationally and theoretically. Various portfolio models such as the single-index model, the multi-index model [1] and the mean-absolute deviation (MAD) model [2] have been proposed. The single-index model reduces the number of parameters for representing the variance-covariances of the stocks by assuming a linear relation for the return on stocks and the return of the market index. The multi-index model then extends the linear relation on a single index to multiple indexes. The MAD model, however, uses the mean absolute deviation from the mean as the risk measure to estimate the nonlinear variance-covariances of the stocks in the mean-variance model. It transforms the portfolio selection problem from a quadratic programming into a linear programming problem. Usually, it is easier to solve a linear problem than a quadratic one [8]. If the returns are normally distributed, then the risk measures in the MAD model becomes proportional to the standard deviation. Hence the corresponding MAD model is then equivalent to the Markowitz mean-variance model. However, the MAD model does not require any specific type of return distributions.

The popularity of downside risk among investors is growing and mean-return-downside risk portfolio selection models seem to oppress the familiar mean-variance approach. The reason for the success of the former models is that they separate return fluctuations into downside risk and upside potential. This is especially relevant for asymmetrical return distributions, for which mean-variance model punish the upside potential in the same fashion as the downside risk. This led Markowitz [4] to propose downside risk measures such as (downside) semivariance to replace variance as the risk measure. Consequently, one observes growing popularity of downside risk models for portfolio selection [6].

The portfolio optimization problem considered in this paper follows the original Markowitz formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates capital among various securities. Assuming that each security is represented by a variable, this is equivalent to assigning a nonnegative weight to each variable. During the investment period, a security generates a random rate of return. The change of capital invested observed at the end of the period is measured by the weighted average of the individual rates of return.

The mean absolute negative deviation from the mean is a half of the mean absolute deviation from the mean, hence the corresponding mean-risk model is equivalent to the MAD model [5]. We model a downside portfolio selection problem as a linear programming problem by taking the mean negative absolute deviation as the risk measure. We compare the result from the model with the results from MAD and mean-variance models.

2. Portfolio Models

Suppose there are n stocks considered for an investment. Let the return on stock i, i = 1, ..., n, be denoted by random variable R_i with mean $\overline{R}_i$. Let X_i be the proportion invested on stock i and we call $P=(X_1,X_2, \ldots , X_n)$, where $\sum_{i=1}^{n} X_i = 1$, a portfolio. Each portfolio P defines a corresponding random variable $R_P = \sum_{i=1}^{n} R_i X_i$ that represents return of the portfolio. The expected return $\overline{R}_P$ of this portfolio:

$$\overline{R}_P = E\left[\sum_{i=1}^{n} R_i X_i\right] = \sum_{i=1}^{n} \overline{R}_i X_i$$

We consider the portfolio optimization problem modeled as a mean-risk bicriteria optimization problem where $\overline{R}_P$ is maximizes and some risk δ_P is minimized.

In order to compare on real-life data the performance of various mean-risk models, one needs to deal with specific investor preferences expressed in the models. One way of modeling risk averse preferences and therefore one of the major approaches to handle bicriteria mean-risk problems is by assuming a trade-off coefficient between the risk and the mean, the so-called risk aversion coefficient [7]. Let λ = the risk aversion factor of the investor satisfying $0 \le \lambda \le 1$. The greater the aversion factor, λ, the more risk aversion the investor has. When $\lambda = 1$, the investor will be extremely conservative because in this case only the risk of his/her investment is considered and no attention is paid to the returns of his/her investment. Conversely, $\lambda=0$ means that the investor is extremely aggressive to pursue the returns of his/her investment, completely ignoring the risk of investment.

2.1 Mean-Variance Portfolio Optimization Model.

Let σ_P be the standard deviation of the portfolio be the risk measure:

$$\sigma_P = \sqrt{E[\{\sum_{i=1}^{n} R_i X_i - (E\sum_{i=1}^{n} R_i X_i)\}^2]} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij} X_i X_j}$$

where σ_{ij} is the covariance of the returns on stock i and j.

The mean-variance model can then be written as a quadratic programming as follows [see 9]

MODEL P1: Mean variance (MV) model

$$\text{Max } Z = (1-\lambda)\sum_{i=1}^{n} \overline{R}_i X_i - \lambda \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} X_i X_j$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$L_i \le X_i \le U_i$, $i = 1,\ldots,n$

$X_i \ge 0$, $i = 1,\ldots,n$

where L_i and U_i are the lower bound and the upper bound on the proportion of stock i respectively.

2.2 Mean Absolute Deviation Optimization Model

Let R_{it} = the realization of random variable R_i during period t (where t = 1, … , T). R_{it} is available from historical data or from some future projection. Konno and Yamazaki assumed that the expected value of the random variable can be approximated by the average derived from these data [2]. That is:

$$\overline{R}_i = E[R_i] \cong \frac{1}{T}\sum_{t=1}^{T} R_{it}$$

The absolute deviation is defined as $\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(R_{it}-\overline{R}_i)X_i\right|$ and is used to replace the term $\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}X_iX_j$ in the mean-variance model. The MAD model can then be expressed as:

MODEL P2: Mean-Absolute Deviation (MAD) model

$$\text{Maximize } (1-\lambda)\sum_{i=1}^{n}\overline{R}_iX_i - \frac{\lambda}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(R_{it}-\overline{R}_i)X_i\right|$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$$L_i \le X_i \le U_i, \quad i = 1,\ldots,n$$
$$X_i \ge 0, \quad i = 1,\ldots,n$$

Model P2 can be transformed to a linear programming optimization model as follows:

Let $d_t = = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(R_{it}-\overline{R}_i)X_i\right|$ denote the absolute deviation of the portfolio return (from the mean) at time t, then the equivalent linear program for the mean-absolute deviation model is:

$$\text{Max } Z = (1-\lambda)\sum_{i=1}^{n}\overline{R}_iX_i - \lambda\frac{1}{T}\sum_{t=1}^{T}d_t$$

Subject to

$$\sum_{i=1}^{n}(\overline{R}_i - R_{it})X_i \le d_t, \quad t = 1, \ldots, T \quad \text{(downside)}$$

$$\sum_{i=1}^{n}(\bar{R}_i - R_{it})X_i \geq -d_t,\quad t=1,\dots,T \text{ (upside)}$$

$$\sum_{i=1}^{n} X_i = 1$$

$L_i \leq X_i \leq U_i,\quad i = 1,\dots,n$

$X_i \geq 0,\quad i = 1,\dots,n$

$d_t \geq 0,\quad t = 1,\dots,T$

2.3. Downside Portfolio Optimization Model

By considering only the return below the mean as the risk and replace the mean absolute deviation with the mean negative deviation the downside risk model can be written as:

MODEL P3 : Mean-Semi Absolute Deviation (MSAD) Model

$$\text{Maximize } (1-\lambda)\sum_{i=1}^{n}\bar{R}_iX_i - \lambda\{\max[\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n}(\bar{R}_i - R_{it})X_i, 0]\}$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$L_i \leq X_i \leq U_i$,

$X_i \geq 0,\ i = 1, 2, \dots, n,$

Let $\bar{d}_t = \max[\sum_{i=1}^{n}(\bar{R}_i - R_{it})X_i, 0]$ denote the absolute deviation of the portfolio return (from the mean) at time t, then the mean-semi absolute deviation model can be transformed to an equivalent linear program as:

$$\text{Maximize } (1-\lambda)\sum_{i=1}^{n}\bar{R}_iX_i - \frac{\lambda}{T}\sum_{t=1}^{T}\bar{d}_t$$

Subject to

$$\sum_{i=1}^{n}(\bar{R}_i - R_{it})X_i \leq \bar{d}_t \quad t=1,\dots,T$$

$$\sum_{i=1}^{n} X_i = 1$$

$L_i \leq X_i \leq U_i$,

$X_i \geq 0,\ i = 1, 2, \dots, n,$

$\bar{d}_t \geq 0 \qquad t = 1,\dots,T$

3. Methodology

We tested our model on 40 stocks chosen at random from stocks listed on the main board of Bursa Malaysia. The historical data of beginning and ending monthly price with dividend yield from 1994 to 2003 for these stocks were used to obtain the annual return for ten years from 1994 to 2003.

Let R_{it} and R_i^t represent the return of stock i in month t and in year t respectively.

$$R_{it} = \frac{P_{i(t+1)} - P_{it} + D_{it}}{P_{it}} \quad \text{and} \quad R_i^t = (1+R_{i1})(1+R_{i2}) \ldots (1+R_{i12}) - 1,$$

where P_{it} and $P_{i(t+1)}$ represent the beginning and ending price respectively and D_{it} is the dividend.

The optimal portfolios for different risk aversion factors and different upperbounds on the proportions were obtained from the downside risk model that is the mean-semi absolute deviation(MSAD) model:

I) MSAD model

$$\text{Maximize } (1-\lambda)\sum_{i=1}^{n} \overline{R}_i X_i \; - \lambda\{\max[\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n}(\overline{R}_i - R_i^t)X_i, 0]\}$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$$0 \le X_i \le U, \quad i = 1, 2, \ldots, n$$

where T = 10 and t = 1,2,…,T represent the ten years from 1994 to 2003 and $\overline{R}_i$ is the average annual return for stock i.

The return of the optimal portfolio were compared with those obtained from the mean-variance model and the MAD model.

II) MV model

$$\text{Max } Z = (1-\lambda)\sum_{i=1}^{n} \overline{R}_i X_i - \lambda \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} X_i X_j$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$$0 \le X_i \le U_i, \quad i = 1, \ldots, n$$

where $\overline{R}_i$ is the expected annual return on stock i and σ_{ij} is the estimated covariance of the annual return on stocks i and j.

III) MAD model

$$\text{Maximize } (1-\lambda)\sum_{i=1}^{n}\overline{R}_i X_i - \frac{\lambda}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(R_i^t-\overline{R}_i)X_i\right|$$

Subject to

$$\sum_{i=1}^{n} X_i = 1$$

$$0 \le X_i \le U, \quad i = 1,\ldots,n$$

where $T = 10$ and $t = 1,2\ldots,T$ represent the ten years from 1994 to 2003. R_i^t is the annual return on stock i estimated from year t and $\overline{R}_i$ is the corresponding average annual return. The models were solved using LINGO optimization software to find optimal solution for the cases of the risk aversion coefficient λ = 0.95, 0.8, 0.6, 0.5, 0.4, 0.2, 0.05 and upper bound U = 0.1,0.2 ,0.3.

4. Numerical Results

The effect of downside risk is analyzed by comparing the return of optimal portfolio from the mean absolute deviation model with the return of portfolio from semi absolute deviation model. The results are presented in tables 1-3.

Table 1: Optimal portfolio return with upper bound U = 0

	Risk aversion coefficient, λ						
	0.05	0.2	0.4	0.5	0.6	0.8	0.95
MSAD	0.880322	0.880322	0.88032	0.880322	0.784905	0.506075	0.80216
MAD	0.880322	0.880322	0.85049	0.729723	0.574451	0.319444	0.62166
MV	0.843519	0.867445	0.75974	0.669187	0.570579	0.440827	0.25915

The return of optimal portfolio from MSAD model is greater than the return resulting from solving MAD model for λ=0.4, 0.5, 0.6, 0.8 and 0.95 the MSAD return is equal to MAD model for λ=0.05 and 0.2. The return of optimal portfolio from MSAD model is better than the return from MV model for all λ.

Table 2: Optimal portfolio return with upper bound U = 0.2

	Risk aversion coefficient, λ						
	0.05	0.2	0.4	0.5	0.6	0.8	0.95
MSAD	1.143708	1.143708	1.143708	1.06203	1.012542	0.536455	1.062028
MAD	1.143708	1.143708	1.012542	0.73453	0.565218	0.536455	0.716509
MV	0.998515	1.062028	0.876051	0.80442	0.694182	0.543691	0.406437

The optimal portfolio using MSAD provides better return than the other two models. Return from MSAD is less than the return in MV model only when λ= 0.8.

Table 3: Optimal portfolio return with upper bound U = 0.3

	Risk aversion coefficient, λ						
	0.05	0.2	0.4	0.5	0.6	0.8	0.95
MSAD	1.27514	1.248213	1.223329	1.223329	1.198587	0.582552	1.223329
MAD	1.27514	1.243373	1.202533	0.791408	0.680677	0.513498	0.681918
MV	0.997221	1.198587	0.894555	0.804037	0.742367	0.576181	0.395898

As the result for upperbound U = 0 in table 1, the return of the optimal portfolio from MSAD model is better than that of the MAD and MV models for all values of risk aversion coefficients.

It can be observed that, in the sense of investment returns, the optimal portfolios from the mean absolute negative deviation or the downside risk optimal portfolio result in higher returns. Thus, the downside risk model is preferable when taking arithmetic means as expected returns of stocks.

5. Conclusion

In this paper, a portfolio selection of stocks with downside risk is modeled as a mean-risk bicriteria portfolio optimization problem. The mean absolute negative deviation of annual return from the average annual return is used the downside risk. The annual returns are calculated using the monthly returns. The portfolio selection problem with 40 stocks were then tested to determine the optimal portfolio. The returns of the optimal portfolios are compared to the performance of the model with the other models.
The comparison shows that the performance of portfolio model with downside risk is better than that of the mean-absolute deviation model and the mean-variance model.

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