THE EXPONENT SET OF COMPLETE ASYMMETRIC 2-DIGRAPHS

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Abstract

A 2-digraph is a digraph whose each of its arcs is colored by either red or blue. The exponent of a 2-digraph D is the smallest positive integer h + k over all possible nonnegative integers h and k such that for each pair of vertices u and v in D there is a walk from u to v consisting of h red arcs and k blue arcs. In this paper, we show that for n ≥ 5 the exponent set of complete asymmetric 2-digraphs on n vertices is E_n = {2,3,4}.

Keywords: 2-digraphs, primitive, exponent.

1. Introduction

Let D be a digraph. We follow the notations and terminologies of digraphs on Brualdi and Ryser [1]. By a walk of length m from a vertex u to a vertex v, we mean a sequence of arcs of the form

$$(v_0,v_1),(v_1,v_2),\ldots,(v_{m-2,}v_{m-1}),(v_{m-1},v_m) \tag{1}$$

where $v_0 = u$ and $v_m = v$. The walk (1) is also denoted by

$$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m.$$

A walk is called *closed* if $u = v$ and is called *open* otherwise. A *path* is a walk with no repeated vertices and a *cycle* is a closed path. A digraph D is *strongly connected* provided for each pair of vertices u and v in D there is a walk from u to v and vice versa. A digraph D is *complete* provided that for each pair of vertices u and v both (u,v) and (v,u) are arcs of D. A strongly connected digraph is *primitive* provided there is a positive integer k such that for each pair of vertices u and v in D there is a walk of length k from u to v. The smallest of such positive integer k is the *exponent* of D and is denoted by exp(D). Wielandt [6] shows that for any primitive digraph of order n, the exp(D) $\leq (n-1)^2 + 1$. Let E_n be the set of positive integers t such that there is a primitive digraph having t as its exponent. Dulmage and Mendelsohn [2] show that for $n \geq 4$, the set E_n is a proper subset of $\{1, 2, \ldots, (n-1)^2 + 1\}$.

By a 2-*colored digraph* or a 2-*digraph* we mean a digraph whose each of its arcs is colored by either red or blue. In a 2-digraph we distinguish a walk of length m by how many red arcs and blue arcs it has. By an (h,k)-walk we mean a walk of length $h+k$ consisting of h red arcs and k blue arcs for some nonnegative integers h and k. For each walk w, $r(w)$ and

$b(w)$ denote respectively the number of red arcs and blue arcs w has. The vector $\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}$ is called the *composition* of w. A 2-digraph D is *strongly connected* provided that its underlying digraph, that is the digraph obtained from D by ignoring its arcs color, is strongly connected. A strongly connected 2-digraph D is *primitive* provided there exist nonnegative integers h and k such that for each pair of vertices in D there is an (h,k)-walk between them. The smallest positive integer $h+k$ over all such nonnegative integers h and k is called the *exponent* of D and is denoted by $\exp(D)$. By a complete asymmetric 2-*digraph* we mean a complete 2-digraph such that whenever (u,v) is a red arc then (v,u) is a blue arc and vice versa.

Let D be a strongly connected 2-digraph and let $C = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be the set of all cycles in D. The *cycle matrix* of D is the 2 by t matrix

$$M = \begin{bmatrix} r(\gamma_1) & r(\gamma_2) & \cdots & r(\gamma_t) \\ b(\gamma_1) & b(\gamma_2) & \cdots & b(\gamma_t) \end{bmatrix}$$

where the ith column of M is the composition of the cycle $\gamma_i, i = 1,2,\cdots,t$. We define $\langle M \rangle$ to be the additive subgroup of $\mathbf{Z}^2$ generated by the columns of M. If the rank$(M) = 2$ the *content* of M, denoted by cont(M), is defined to be the greatest common divisor of the determinant of 2 by 2 submatrices of M. If rank$(M) = 1$, then we define the content of M to be 0. The following result, due to Fornasini and Valcher (see Theorem 2 of [3]), gives algebraic characterization of primitive 2-digraphs.

Theorem 1. *Let D be a strongly connected 2-digraph whose cycle matrix M has rank 2. Then the followings are equivalent*:

(a) *D is primitive,* (b) $\langle M \rangle = \mathbf{Z}^2$, (c) cont$(M) = 1$.

Shader and Suwilo [5] show that the largest exponent for primitive 2-digraphs on n vertices lies in the interval $[(n^3-5n^2)/2, (3n^3 + 2n^2-2n)/2]$. Olesky et al. [4] generalize the notion of 2-digraphs to that of *multicolored digraphs* for positive integer $k \geq 2$ and show that the largest exponent of primitive *multicolored* k-digraphs is of order $\Theta(n^{k+1})$. In this paper, for $n \geq 5$ we show that the exponent set E_n of asymmetric complete 2-digraph of order n is $E_n = \{2,3,4\}$.

2. Notes on Exponents of 2-Digraphs

We give several comments on exponents of 2-digraphs. Let D' be a 2-digraph obtained from D be replacing each red arc by blue arc and vice versa. This implies every (h,k)-walk in D' corresponds to a (k,h)-walk in D, and hence $\exp(D)$ equals to $\exp(D')$. Let D be a primitive asymmetric complete 2-digraph. Suppose that the exponent of D is attained by (h,k)-walks. Hence for any pair of vertices u and v in D there is an (h,k)-walk from u to v

and there is an (h,k)-walk from v to u. Since D is asymmetric complete 2-digraph, for every pair of vertices u and v in D there is a (k,h)-walk from u to v and there is a (k,h)-walk from v to u.

3. The Exponent Set of Asymmetric Complete 2-digraphs

Let D be a 2-digraph on n vertices. Let v be any vertex in D. The *red indegree* of the vertex v, denoted by rid(v), is the number of red arcs incident to v. The *red outdegree* of the vertex v, denoted by rod(v), is the number of red arcs incident from v. The *blue indegree* and *outdegree* are defined similarly. We define the *exponent set*, denoted by E_n, of a 2-digraph D on n vertices to be the set of all positive integers t such that there is a 2-digraph on n vertices having t as its exponent. In the following we show that for $n \geq 5$ the exponent set of asymmetric complete 2-digraphs on n vertices is $E_n = \{2, 3, 4\}$. We first state a result on exponents of asymmetric complete 2-digraphs.

Theorem 2. *Let D be a complete asymmetric 2-digraph on $n \geq 3$ vertices. Then* $\exp(D) \leq 4$.

Proof. Let u and v be vertices in D. We show that there exists a (2, 2)-walk from u to v. Since D has $n \geq 3$ vertices, there is a vertex x in D where $x \neq u, v$. If $u = v$, then the walk

$$u \rightarrow x \rightarrow u \rightarrow x \rightarrow u$$

is a (2, 2)-walk from u to itself. Assume now that $u \neq v$.
Let

$$V_R(u) = \{x \in V : x \neq v \text{ and the arc } (u, x) \text{ is a red arc}\}$$

and

$$V_R(u) = \{y \in V : y \neq v \text{ and the arc } (u, y) \text{ is a blue arc}\}.$$

If there is an $x \in V_R(u)$ such that the arc (x,v) is a blue arc, then the walk

$$u \xrightarrow{r} x \xrightarrow{b} v \xrightarrow{r} x \xrightarrow{b} v$$

is a (2,2)-walk from u to v in D. Similarly, if there is a $y \in V_B(u)$ such that (y,v) is a red arc, then the walk $u \xrightarrow{b} y \xrightarrow{r} v \xrightarrow{b} y \xrightarrow{r} v$ is a (2,2)-walk form u to v. Hence, we assume that (x,v) is red for all $x \in V_R(u)$, and (y,v) is blue for all $y \in V_B(u)$. Since $n \geq 3$, either $V_R(u) \neq \varnothing$ or $V_B(u) \neq \varnothing$. We consider three cases.

Case 1: $V_R(u) \neq \varnothing$ and $V_B(u) = \varnothing$.

Let $x \in V_R(u)$. If the arc (u,v) is red, then the walk $u \xrightarrow{r} v \xrightarrow{b} x \xrightarrow{b} u \xrightarrow{r} v$ is a (2,2)-walk from u to v. Suppose the arc (u,v) is a blue arc. The cycles $u \rightarrow x \rightarrow u$, $u \rightarrow v \rightarrow u$, $x \rightarrow v \rightarrow x$, $u \rightarrow x \rightarrow v \rightarrow u$, and $x \rightarrow u \rightarrow v \rightarrow x$ have composition $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}$, and $\begin{bmatrix}0\\3\end{bmatrix}$ respectively. Since D is 2-primitive and the content of $\begin{bmatrix} 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 & 3 \end{bmatrix}$ is not 1, $|V_R(u)| > 1$. Let x_1 and x_2 be distinct vertices in $V_R(u)$. If the arc (x_1, x_2) is a red arc, then the walk $u \xrightarrow{b} v \xrightarrow{b} x_1 \xrightarrow{r} x_2 \xrightarrow{r} v$ is a (2,2)-walk from u to v. If the arc (x_1, x_2) is a blue arc, then the walk $u \xrightarrow{b} v \xrightarrow{b} x_2 \xrightarrow{r} x_1 \xrightarrow{r} v$ is a (2,2)-walk from u to v.

Case 2: $V_R(u) = \varnothing$ and $V_B(u) \neq \varnothing$.
An argument similar to that of Case 1 shows that for each pair of vertices u and v in D there exists a (2,2)-walk form u to v.

Case 3: $V_R(u) \neq \varnothing$ and $V_B(u) \neq \varnothing$.
Choose $x \in V_R(u)$ and $y \in V_B(u)$. If the arc (u,v) is a red arc, then the walk

$$u \xrightarrow{r} v \xrightarrow{b} x \xrightarrow{b} u \xrightarrow{r} v$$

is a (2,2)-walk from u to v. If the arc (u,v) is a blue arc, then the walk

$$u \xrightarrow{b} v \xrightarrow{r} y \xrightarrow{r} u \xrightarrow{b} v$$

is a (2,2)-walk from u to v.

Therefore, for each pair of vertices u and v, there exists a (2,2)-walk from u to v. Hence, we conclude that $\exp(D) \leq 4$. ■

We note that Theorem 2 implies that $E_n \subseteq \{1,2,3,4\}$. Since D is asymmetric, for each pair of vertices u and v if the arc (u,v) is red then the arc (v,u) is blue and vice versa. This implies that $\exp(D) \geq 2$. Hence $E_n \subseteq \{2,3,4\}$. The following theorem gives sufficient condition for a complete asymmetric 2-digraph to have exponent equals 2.

Proposition 3. *Let D be a complete asymmetric 2-digraph on $n \geq 4$. Suppose D has a vertex v with* $\mathrm{rid}(v) = n - 1$ *and for every vertex $u \neq v$ in D we have* $\mathrm{rid}(u) \geq 1$. *Then the* $\exp(D) = 2$.

Proof. Let the vertex set of D be $V = \{v_1, v_2, \ldots, v_n\}$ and without loss of generality we assume that $\mathrm{rid}(v_1) = n - 1$. We show that for each pair of vertices v_i and v_j in D there is a (1,1)-walk from v_i to v_j. Clearly, for each vertex v_i the walk $v_i \rightarrow v_j \rightarrow v_i$ is a (1,1)-walk. We note that for every pair of vertices v_i and v_j where $v_i, v_j \neq v_1$, the walk $v_i \rightarrow v_1 \rightarrow v_j$ is a (1,1)-walk from v_i to v_j. Now assume that $v_i = v_1$. Since $\mathrm{rid}(v_j) \geq 1$, then for each v_j there is a vertex v_k such that (v_k, v_j) is a red arc. This implies $v_1 \rightarrow v_k \rightarrow v_j$ is a (1,1)-walk. Similar argument shows that $v_j \rightarrow v_k \rightarrow v_1$ is a (1,1)-walk. ■

The following proposition gives a class of complete asymmetric 2-digraph with exponent equals 3.

Proposition 4. *Let D be a complete digraph on $n \geq 5$ vertices $v_1, v_2, ..., v_n$. Let D be colored such that the red arcs of D are at least the arcs of the form (v_i,v_2) for all $i \neq 3$, (v_i,v_3) for all $i \neq 1$,* (v_n,v_i) for all $i = 2, ..., n-1$, *the arcs (v_1,v_n),(v_{n-1},v_1), (v_3,v_1), and color the rest arcs in D with either red and blue such that D is a complete asymmetric 2-digraph. Then the* exp(D) = 3.

Proof. We first note that there are no (2,0)-walks from v_2 to v_3 and there are no (1,1)-walks from v_n to v_3. Hence the exp(D) $\geq$ 3. We show that for each pair of vertices in D there is a (2,1)-walk between them. This will imply that the exp(D) = 3.

Note that for any vertex $v_i \neq v_2, v_3$ and any vertex v_j, the following walks $v_i \to v_2 \to v_{n-1} \to v_1$, $v_i \to v_2 \to v_3 \to v_n$, and $v_i \to v_2 \to v_n \to v_j$ are (2,1)-walk. These show that for any pair of vertices $v_i \neq v_2, v_3$ and v_j there is a (2,1)-walk from v_i to v_j. If $v_i = v_2$, then the walks $v_2 \to v_3 \to v_{n-1} \to v_1$, $v_2 \to v_3 \to v_{n-1} \to v_n$, and $v_2 \to v_3 \to v_n \to v_j$, $2 \leq j \leq n-1$, are (2,1)-walks with initial vertex v_2. Finally if $v_i = v_3$, then the walk $v_3 \to v_1 \to v_{n-1} \to v_j$ for $1 \leq j \leq 3$ and the walks $v_3 \to v_1 \to v_2 \to v_j$ for $4 \leq j \leq n$ are (2,1)-walks with initial vertex v_3. ■

Proposition 5. *Let D be a complete asymmetric 2-digraph on $n \geq 3$ vertices. If D has a vertex u with* rid(u) = $n-1$ *and has a vertex v with* rod(v) = $n-1$, *then the* exp(D) = 4.

Proof. By Theorem 2, it suffices to show that exp(D) $\geq$ 4. Consider the vertex u with rid(u) = $n-1$ and the vertex v with rod(v) = $n-1$. Since all arcs with terminal vertex u are red, there are no (2,0)-walks from v to v. Since rid(u) = $n-1$ and rod(v) = $n-1$, all walks of length 2 from u to v are (0,2)-walks. Hence there are no (1,1)-walks from u to v. Therefore, the exp(D) $\geq$ 3.

Since all arcs with terminal vertex v are blue arcs, there are no (3,0)-walks with terminal vertex v. More over since all arcs with initial vertex u are blue arcs, there are no (2,1)-walks form u to v. Hence the exp(D) $\geq$ 4. ■

We now present the main result.

Theorem 6. *Let D be a complete asymmetric 2-digraph on $n \geq 5$ vertices. Then the exponent set of D is* $E_n = \{2,3,4\}$.

Proof. Since D is a complete asymmetric 2-digraph, if (u,v) is a red arc the (v,u) is a blue arc and vice versa. This implies the exp(D) $\geq$ 2. The sequence of Propositions 3, 4, and 5, and Theorem 2 imply that E_n = {2, 3, 4}. ■

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