LINEAR EQUATIONS OVER COMMUTATIVE RINGS AND DETERMINANTAL IDEALS

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Abstract

This paper discusses about necessary and sufficient condition for linier equations over commutative rings and relation to determinantal ideals.

Keywords: system, vector, Crame's rule.

1. Introduction

In this paper, *R* will denote a commutative ring with unit, a system of linear equation over *R* given by

$$(S): Ax = B \qquad (1)$$

where $A=[a_{ij}]$ is an $(m\times n)$ - matrix over *R*, $x=[x_1,\ldots,x_n]^t$ and $B=[b_1,\ldots,b_m]^t$ are column vectors with over *R*.

Equation in (1) have a solution, if there exist a vector ξ in R^n such that $A\xi = B$. For *A* is a square matrix of size *n* Crame's rule tell us about the determinantal approach, that is if det(*A*) is a unit, then we may solve uniquely for x_i, $1\le i\le n$.

If we extend *A* be an $(m\times n)$ - matrix and we defined $I_t(A)$ as the ideal generated by all the determinant of the $t\times t$ sub matrices of *A* - $I_t(A)$ was the *t–th* determinantal ideal of *A*, and $I_t(A\mid B)$ is determinantal ideal of augmented matrix [A|B], then next theorem describes necessary condition for $Ax=B$ to have solution, that is if system $(S): Ax = B$ has solution then $I_t(A\mid B)=I_t(A)$ for all $t\in\mathbf{Z}$.

This condition is not sufficient to assure a solution of $Ax = B$. Camion, Levy and Mann have given an example where the determinantal ideal *A* and augmented matrix [A|B] agree for all *t*, but $Ax=B$ has no solution.

Which is due Camion, Levy and Mann, we give sufficient condition for $Ax=B$ to have solution. In this case, we will let $I_t(A\mid B)^*$ denote the ideal generated by all determinant of $t\times t$ sub matrices of [A|B], and we can always assume $m\le n$ to considering solution to $Ax=B$. That is $rank(A)=m$, if there exist ideal μ in R and nonzero devisor α, such that $\mu I_m(A\mid B)^*\subseteq<\alpha>\subseteq\mu I_m(A)$, then the equation $Ax=B$ has a solution. The origin of this result may possibly be traced to Steinitz and his interest in solving system of linear equation over ring of algebraic integers or more generally, Dedekind domain.

Another application is given by this following result. If $I_t(A)=R$, than $Ax=B$ is solvable. For $B=0$ a system of linear equation become $Ax=0$ of course, this system always possesses the trivial solution $x=0$. McCoy tell us this system has non-trivial solution if and only if McCoy rank of A less than n.

In this paper we want to give sufficient and necessary condition for system of linear equation has a solution with the local property.

2. Localization

For any prime ideal P in R we construct sets and $R\times S=\{(r,s)\mid r\in R,\ s\in S\}$, we define a relation ~ or $R\times S$ as follow $(r,s)\sim(\overline{r},\overline{s})$ if only if $t(r\overline{s}^{'}-\overline{r}^{'}s)=0$ for some $t\in P^c$

It is easy to check that ~ is an equivalence relation on $R\times S$. We will let $\frac{r}{s}$ denote the equivalence class of (r, s) in $R\times S$.

Set $R_p=\{\frac{r}{s}\mid(r,s)\in R\times S\}=\{\frac{r}{s}\mid r\in R,\ s\notin P\}$

Addition and multiplication on R_P are defined by following equations :

$$\frac{r}{s}+\frac{r^{'}}{s^{'}}=\frac{(rs^{'}+sr^{'})}{ss^{'}}$$

$$(\frac{r}{s}).(\frac{r^{'}}{s^{'}})=\frac{rr^{'}}{ss^{'}}$$

We can check that both of these operation are well defined and R_P with the structure of a commutative ring. The ring R_P is a local ring that is ring has precisely one maximal ideal and called the localization of R at P. For any prime ideal P in R the system of equation over R_P obtained from $(S){:}\,Ax=B$ by replacing each coefficient by its image in R_P via the homomorphism $\theta{:}\,R\rightarrow R_P$ will be denoted by (S_p) that given by

$(S_p){:}\,A_px=B_p$ where $A_p=(\frac{a_{ij}}{1})$ and $B_p=(\frac{b_1}{1},\ldots,\frac{b_m}{1})^t$

The following result shows that the existence of a solution for (S) is a local property.

Theorem 1: *The following statements are equivalent*

(a) *The system* (S) : $Ax = B$ *has a solution over* R

(b) *The system* (S_p) : $A_p x = B_p$ *has a solution over local* R_P *for each prime ideal P or R*

(c) *The system* (S_m) : $A_m x = B_m$ *has a solution over local ring* R_m *for each maximal ideal M of R*

Proof:

(i) → (ii) → (iii) is easy to proof

(iii) → (i) Assume that for every maximal *M* in *R* the system (S_M) has a solution given by

$$x_j(M) = \frac{\alpha_j}{s(m)} \qquad ,1 \le j \le n$$

in this case assumption that $x_1(m), \ldots, x_n(m)$ have the same denominator $s(m) \notin M$. Then for every *M* one has

$$\sum_{j=1}^{n} a_{ij} x_j \frac{\alpha_j(m)}{s(m)} = b_i \qquad ,1 \le i \le m$$

so there exist element $t_i(m) \notin M$ such that

$$t_i(m)\left(\sum_{j=1}^{n} a_{ij}\alpha_j(m)\right) = t_i(m)s(m)b_i \qquad ,1 \le i \le m$$

setting $t(m) = \prod_{i=1}^{n} t_i(m)$ one obtains.

$$t_i(m)\left(\sum_{i=1}^{m} a_{ij}\alpha_j(m)\right) = t_i(m)s(m)b_i \qquad ,1 \le i \le m$$

Now, since $t(m).s(m) \notin M$, ideal generated by elements *t*(*m*) . *s*(*m*) is not contained in any maximal ideal *M* and so $t(m).s(m)$ is *R*. It follows that there exist finitely many maximal ideal $m_1, \ldots, m_p$ and element $\lambda_1, \ldots, \lambda_p$ in *R* such that

$$1 = \sum_{k=1}^{p} \lambda_k t(m_k) s(m_k)$$

Finally, the element $x_1, \ldots, x_n$ given by

$$x_j = \sum_{k=1}^{p} \lambda_k t(m_k)\, s(m_k) x_j$$
$$= \sum_{k=1}^{p} \lambda_k t(m_k)\, \alpha_j(m_k) \qquad , 1 \le j \le n$$

are a solution of system (*S*) ■

In a system of linear equation as above we concerned only with the matrices *A*, (*A* | *B*) and A_p for *P* is a prime ideal in *R*. The following properties we will use to find the necessary condition for system so that has a solution in the particular class ring.

Nakayama's Lemma : *Let I finitely generated ideal in a local ring. If* r *is the smallest number of generators of* μ, *then any set of generator of* I *contains a set of* r *generator of* I.

Theorem 2: *Let R be a local ring and* (S) : $Ax = B$ *a system of linear equation over R.. The system (S) has a solution in R if satisfies the following two condition for every* $t \ge 0$

(i) $I_t(A) = I_t(A \mid B)$

(ii) *Either* $I_t(A) = 0$ or $I_t(A)$ *is an ideal generated by a nonzero divisor of R*

Proof:

Let *r* be the integer such that $I_t(A) \ne 0$ and $I_t(A \mid B) = 0$. Since $I_t(A)$ is generated by all determinant of $t \times t$ sub matrices of *A* and by the hypothesis (ii), it is a principal ideal generated by a nonzero divisor in the local ring *R*.

From Nakayama's lemma that $I_t(A)$ is generated by an determinant of $t \times t$ sub matrices of *A*. We can assume with no loss of generality that $\Delta(i_1, \ldots, i_r : j_1, \ldots, j_r) = \Delta(1, \ldots, r : 1, \ldots, r)$. Let $\lambda = \Delta(1, \ldots, r : 1, \ldots, r)$. Then we have

$$A' = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix}$$

Now we construct the system (S') given by

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_{ij} \qquad , 1 \le i \le r$$

by multiplying by cofactor matrix $(A_{k,j})$ of A' on the left of system (S')

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \qquad , 1 \le i \le j$$

$$\sum_{j=1}^{n} a_{ij}\, x_j + \sum_{t=1}^{n-r} a_{i,r+t}\, x_{r+t} = b_i \qquad ,1 \le i \le r$$

$$\sum_{j=1}^{r}\sum_{k=1}^{r} A_{k,i}\, a_{ij} x_j + \sum_{t=1}^{n-r}\sum_{k=1}^{r} A_{k,i} a_{i,r+t}\, x_{r+t} = A_{k,i} b_i$$

$$\lambda x_i + \sum_{t=1}^{n-r}\left(\sum_{k=1}^{r} A_{k,i} a_{i,r+t}\right) x_{r+t} = \sum_{k=1}^{n} A_{k,i} b_i \qquad , 1 \le i \le r$$

We can be also written in the form

$$\lambda x_i + \sum_{t=1}^{n-r}\begin{vmatrix} a_{11} & \dots & a_{1,i-1} & a_{1,r+t} & a_{i,i+1} & \dots & a_{1r} \\ \vdots & & & & & & \\ a_{r1} & \dots & a_{r,i-1} & a_{r,r+t} & a_{r,i+1} & \dots & a_{rr} \end{vmatrix} x_{r+t}$$

$$= \begin{vmatrix} a_{11} & \dots & a_{1,i-1} & b_1 & a_{i,i+1} & \dots & a_{1r} \\ \vdots & & & & & & \\ a_{r1} & \dots & a_{r,i-1} & b_r & a_{r,i+1} & \dots & a_{rr} \end{vmatrix}$$

so that for $1 \le i \le r$

$$\lambda x_i = \sum_{k=1}^{r} A_{k,i} b_i - \sum_{t=1}^{n-r}\left(\sum_{k=1}^{r} A_{k,i} a_{k,r+t}\right) x_{r+t}$$

Since λ generates the ideal $I_r(A) = I_r(A \mid B)$, then the last expression that the system $(S^{'})$ has a solution. The last we shall check that the solutions of $(S^{'})$ are also solutions of (S). For this we shall show that for $1 \le s \le n-r$ the solution of $(S^{'})$ satisfy the equation.

$$a_{r+s,1}\, x_1 + \dots + a_{r+s,n} x_n = b_{r+s}$$

Since λ is a non-zero divisor, the equation become

$$\sum_{j=1}^{n} \lambda a_{r+s,j}\, x_j = \lambda b_{r+s}$$

or equivalenty

$$\sum_{j=1}^{r} \lambda a_{r+s,j}\, x_j + \sum_{t=1}^{n-r} \lambda a_{r+s,r+t}\, x_{r+t} = \lambda b_{r+s}$$

so that

$$\sum_{j=1}^{r}\lambda a_{r+s,j}\,x_j=\sum_{j=1}^{r}a_{r+s,j}\left(-\sum_{t=1}^{n-r}\left(\sum_{k=1}^{r}A_{k,j}a_{k,r+t}\right)x_{r+t}+\sum_{k=1}^{r}A_{k,j}b_k\right)$$
$$+\sum_{k=1}^{r}\lambda a_{r+s,r+t}\,x_{r+t}$$
$$=\sum_{t=1}^{n-r}\left(\sum_{j=1}^{r}a_{r+s,j}\left(-\sum_{k=1}^{r}A_{k,j}a_{k,r+t}\right)+\lambda a_{r+s,r+t}\right)x_{r+t}$$
$$+\sum_{j=1}^{r}a_{r+s,j}\left(\sum_{k=1}^{r}A_{k,j}b_k\right)$$

For $1\le t\le n-r$, the coefficient of x_{r+t} is zero, since determinant of the $(r+1)\times(r+1)$ - sub matrix of A obtained by adding A' with $(r + t)$ column and $(r + s)$ row of A . So

$$\sum_{j=1}^{r}\lambda a_{r+s,j}\,x_j=\sum_{j=1}^{r}a_{r+s,j}\left(\sum_{k=1}^{r}A_{k,j}b_k\right)$$

so the concludes that the solutions of (S') are also solutions of (S) as desired ■

Proof of the theorem at above with hypothesis or R is a local ring with fact that, if $a_1 \ldots a_n$ is a set of generator of a principle ideal I of a local ring, then I is generated by some a_i .

However, the result can be generated to the nonlocal case by using the concept of flat ideal of a ring.

Defenition: *Ideal I of R that finitely generated is flat if and only if for any prime ideal P of R the ideal* I_p *of* R_p *is* either $\langle 0\rangle$ or generated by a nonzero divisor.

Theorem 3: *Let* $(S): Ax = B$ a system of linear equations over the ring *R*. System (S) has a solution if the following statements hold for every $t \ge 0$

(i) $I_t(A) = I_t(A \mid B)$

(ii) $I_t(A)$ is a flat ideal of *R*.

Proof:

By the property (ii) of determinantal ideals, the ideal I_P of R_P for any prime ideal P is $\langle 0\rangle$ or generated by a nonzero divisor. It is mean the system (S_P) has a solution in R_P, and by theorem 1, system (S) has a solution in *R*. v

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