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EXPONENTS OF PRIMITIVE GRAPHS CONTAINING TWO DISJOINT ODD CYCLES

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Abstract. A connected graph G is primitive provided there exists a positive integer k such that for each pair of vertices u and v in G there is a walk of length k connecting u and v. The smallest of such positive integer k is the exponent of G. A primitive graph is said to be odd primitive graph if it has an odd exponent. It is known that if G is an odd primitive graph then G contains two disjoint odd cycles. This paper discusses exponents of a class of primitive graphs containing of exactly two disjoint odd cycles. For such graphs we characterize the odd and even primitive graphs.

1. Introduction

We discuss exponents of connected graphs consisting of two odd disjoint cycles connected by a path. We follow notation and terminologies for graphs in [1]. In particular a *walk* of length m connecting vertices u and v is a sequence of edges of the form

$$\{u = v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{m-1}, v_m = v\}.$$

A walk w connecting vertices u and v is abbreviated by a uv-walk or w_{uv} and its length is denoted by $\ell(w_{uv})$. By a *uv-path* we mean a uv-walk with no repeated vertices except possibly $u = v$. A uv-walk is open provided $u \neq v$ and is closed otherwise. A *cycle* is a closed path and a *loop* is a cycle of length 1. The *distance* of vertices u and v is the length of the shortest uv-path. The *diameter* of a graph G is defined to be

$$\operatorname{diam}(G) = \max_{u,v \in G} \{d(u, v)\}.$$

A graph G is connected provided for each pair of vertices u and v in G there is a uv-walk connecting u and v. A connected graph is *primitive* provided there is a positive integer k such that for each pair of vertices u and v in G we can find a uv-walk of length k. The smallest of such positive integer k is the *exponent* of G and is denoted by $\exp(G)$. A primitive graph is said to be an *odd primitive graph* if G has odd exponent and is an *even primitive graph* otherwise. The following proposition gives necessary and sufficient conditions for primitivity (see [1]) of connected graphs.

Proposition 1.1. *Let G be a connected graph. The graph G is primitive if and only if G has cycles of odd length.*

A lot of research has been done on exponents of graphs. Shao [4] proved that for primivite graphs G on n vertices $\exp(G) \leq 2n-2$. Liu et.al [3] showed that for loopless primitive graphs on n vertices $\exp(G) \leq 2n-4$. Suwilo and Mardiningsih [5] give a bound of exponents of primitive graphs G in term of the length of the smallest odd cycle in G. Fuyi et.al [2] show that if G is an odd primitive graph, then G contains two odd disjoint cycles. This paper discusses a class of primitive graphs containing two odd disjoint cycles and characterizes them as an odd primitive graph or even primitive graph.

2. Facts on primitive graphs

In this section we present several results on exponents of primitive graphs that will be useful in dicussing our main results. Let G be a primitive graph and let C be a cycle of smallest odd length s. Let u be a vertex in G but not in C and let p_{ux} be a shortest path that connects the vertex u and a vertex x in C, and define

$$\ell = \max_{u \in G \setminus C, x \in C} \{\ell(p_{ux})\}.$$

Suwilo and Mardiningsih [5] proved the following result.

Theorem 2.1. *Let G be a primitive graph with smallest odd cycle of length s. Then*

$$\exp(G) \leq s + 2\ell - 1.$$

Proof. See [5] $\square$

Let G be a connected graph on n vertices consisting of a cycle $C : v_1 - v_2 - \cdots - v_s - v_1$ of length s and a path $P : v_s - v_{s+1} - v_{s+2} - \cdots - v_n$ of length $n - s$ which intersects C on the vertex v_s. The graph G is called (v_s, v_n)-lollipop. Shao showed that the graph attain the upper bound $2n - 2$ is a (v_1, v_n)-lollipop while Liu et.al showed that the graph attain the upper bound $2n - 4$ is a (v_3, v_n)-loolipop. The following corollary gives formula for exponents of (v_s, v_n)-lollipops.

Corollary 2.1. *Let G be a (v_s, v_n)-lollipop with $s \leq n$ is odd. Then* $\exp(G) = 2n - s - 1$.

Proof. Since $\ell = n - s$, Theorem 2.1 implis that $\exp(G) \leq s + 2(n - s) - 1 = 2n - s - 1$. It remains to show that $\exp(G) \geq 2n - s - 1$. We note that the shortest closed walk of odd length that connects the vertex v_n to itself is of length $2n - s$. This implies there is no closed walk of length $2n - s - 2$ that connects v_n to itself. Hence $\exp(G) \geq 2n - s - 1$. □

Corollary 2.2. *Let C be a cycle of odd length s. Then* $\exp(C) = s - 1$.

Proof. A cycle C of length n can be considered as a (v_n, v_n)-lollipop. In this case we have and $s = n$, hence Corollary 2.1 implies that $\exp(C) = s - 1$. □

3. Exponents of primitive graphs containing two disjoint cycles

In this section we explore the exponents of a special type of graphs containing two disjoint odd cycles. Let C_1 and C_2 be two disjoint cycles of odd length s_1 and s_2 respectively. Let P be a path of length ℓ_P with one end vertex on C_1 and the other end vertex on C_2. A connected graph G consisting of two disjoint cycles C_1 and C_2 of length s_1 and s_2 connected by a path P of length ℓ_P is called an (s_1, ℓ_P, s_2)-*barbel*. For simplicity, we assume that $s_1 \leq s_2$ and let $v_1$be the vertex in common to C_1 and P, and v_2 be the vertex in common to C_2 and P. Notice that the diameter of an (s_1, ℓ_P, s_2)-barbel is

$$\mathrm{diam}(G) = \frac{1}{2}(s_1 + s_2) + \ell_P - 1.$$

For each vertex $u \in C_1$, let p_{uv_1} be the shortest uv_1-path and let p'_{uv_1} be the uv_1-path of length $s_1 - \ell(p_{uv_1})$ that lies on C_1. Similarly for each $u \in C_2$ let p_{uv_2} be the shortest uv_2-path and p'_{uv_2} be the uv_2-path of length $s_2 - \ell(p_{uv_2})$ that lies on C_2. The following result characterizes (s_1, ℓ_P, s_2)-barbels of even exponents.

Lemma 3.1. *Let G be an (s_1, ℓ_P, s_2)-barbel with s_1 and s_2 are odd and $s1 \leq s_2$. If* $\mathrm{diam}(G) \leq s_2 - 1$, *then* $\exp(G) = s_2 - 1$.

Proof. Since C_2 is a subgraph of G and the $\exp(C_2) = s_2 - 1$, then $\exp(G) \geq s_2 - 1$. It remains to show that for each pair of vertices u and v in G there is a uv-walk of length exactly $s_2 - 1$.

Case 1. Both vertices $u, v \in C_1$ or $u, v \in C_2$. If u and v both lies on cycle C_1, then Corollary 2.2 implies that there is uv-walk of length $s_1 - 1$. This walk can be extended to a uv-walk of length exacly $s_2 - 1$. Similarly, if u and v both lies on the cycle C_2, Corollary 2.2 implies there is a uv-walk in G of length exactly $s_2 - 1$.

Case 2. The vertex $u \in C_1$ and the vertex $v \in C_2$. Assume that the $\mathrm{diam}(G)$ is obtained from the path p_{xy} consisting of the path p_{xv_1}, the path P, and the path p_{v_2y}, where the vertex $x \in C_1$ and the vertex $y \in C_2$. Notice that we can choose x and y such that both vertices u and v lie on the path p_{xy}. Let p_{uv} be the shortest uv-path and assume that $\ell(p_{uv})$ is odd, since if $\ell(p_{uv})$ is even we can extend p_{uv} to a uv-walk of length $s_2 - 1$ and hence we are done. Consider the path p'_{uv} consisting of the path p'_{uv_1}, the path P and the path p_{v_2v}. If $\ell(p_{ux}) < \ell(p_{vy})$, then

$$\begin{aligned}
\ell(p'_{uv}) &= \ell(p_{ux}) + \ell(p'_{xv_1}) + \ell_P + \ell(p_{v_2v}) \\
&= \ell(p'_{xv_1}) + \ell_P + \ell(p_{v_2y}) + \ell(p_{ux}) - \ell(p_{vy}) \\
&= \frac{1}{2}(s_1 + s_2) + \ell_P + \ell(p_{ux}) - \ell(p_{vy}) \leq \mathrm{diam}(G) \leq s_2 - 1
\end{aligned}$$

is even. Since $\ell(p'_{uv})$ is even, the path p'_{uv} can be extended to a uv-walk of length $s_2 - 1$.

Assume now that $\ell(p_{ux}) > \ell(p_{vy})$. Consider the path p'_{uv} consisting of the path p_{uv_1}, the path P, and the path p'_{v_2v}. Then

$$\begin{aligned}\ell(p'_{uv}) &= \ell(p_{vy}) + \ell(p'_{yv_2}) + \ell_P + \ell(p_{uv_1}) \\ &= \ell(p'_{yv_2}) + \ell(p_{v_1x}) + \ell_P + \ell(p_{vy}) - \ell(p_{ux}) \\ &= \frac{1}{2}(s_1 + s_2) + \ell_P + \ell(p_{vy}) - \ell(p_{ux}) \leq \operatorname{diam}(G) \leq s_2 - 1\end{aligned}$$

is even. We can extend p'_{uv} to a uv-walk of length $s_2 - 1$.

Now assume that $\ell(p_{ux}) = \ell(p_{vy})$. If $\operatorname{diam}(G)$ is even, then $\ell(p_{uv})$ is even. Hence p_{uv} can be extended to uv-walk of length $s_2 - 1$. If $\operatorname{diam}(G)$ is odd, then $\operatorname{diam}(G) < s_2 - 1$. Consider the path p'_{uv} consisting of the path p_{uv_1}, the path P and the path p_{v_2v}.Then

$$\begin{aligned}\ell(p'_{uv}) &= \frac{1}{2}(s_1 + s_2) + \ell_P + \ell(p_{vy}) - \ell(p_{ux}) \\ &= \frac{1}{2}(s_1 + s_2) + \ell_P \leq s_2 - 1.\end{aligned}$$

is even. Hence p'_{uv} can be extended to uv-walk of length $s_2 - 1$.

Case 3. Both vertices $u, v \in P$**.** Assume $\ell(p_{uv})$ is odd and $\ell(p_{uv_1}) < \ell(p_{vv_1})$. Consider the uv-walk w_{uv} consisting of the path p_{uv_1}, the cycle C_1, and the path p_{v_1v}. Then $\ell(w_{uv})$ is even. Since $\operatorname{diam}(G) \leq s_2 - 1$, hence $s_1 + 2\ell_P \leq s_2$. This implies $\ell(w_{uv}) = s_1 + 2\ell(p_{v_1u}) + \ell(p_{uv}) < s_1 + 2\ell_P \leq s_2$. We now have $\ell(w_{uv})$ is even and $\ell(w_{uv}) \leq s_2 - 1$. Hence w_{uv} can be extended to a uv-walk of length exactly $s_2 - 1$.

Case 4. The vertex $u \in C_1$ **or** $u \in C_2$ **and** $v \in P$**.** Assume that $u \in C_1$ and the length of the path p_{uv}, consisting of the path p_{uv_1} and the path p_{v_1v}, is odd. Then the path p'_{uv} consisting of the path p'_{uv_1} and the path p_{v_1v} is of even length and $\ell(p'_{uv}) < s_1 + \ell_P \leq s_2 - \ell_P < s_2 - 1$. Hence p'_{uv} can be extended to a uv-walk of length $s_2 - 1$.

Now assume that $u \in C_2$ and the length of the shortest path p_{uv} is odd. Hence the length of the path p'_{uv} consisting of the path p'_{uv_2} and the path p_{v_2v} is even. If $\ell(p'_{uv}) \leq s_2 - 1$, then we are done. So assume that $\ell(p'_{uv}) > s_2 - 1$. This implies $\ell(p_{vv_2}) > \ell(p_{v_2u})$. Consider the walk w'_{vu} consisting of the path p_{vv_1}, the cycle C_1, the path P, and the path p_{v_2u}. Then $\ell(w'_{uv})$ is even. Since $\ell(p_{vv_2}) > \ell(p_{v_2u})$,

$$\begin{aligned}\ell(w'_{uv}) &= s_1 + \ell_P + \ell(p_{v_1v}) + \ell(p_{v_2u}) < s_1 + \ell_P + \ell(p_{v_1v}) + \ell(p_{vv_2}) \\ &= s_1 + 2\ell_P \leq s_2.\end{aligned}$$

Since $\ell(w'_{uv})$ is even, $\ell(w'_{uv}) \leq s_2 - 1$. Now we can extend w'_{uv} to a uv-walk of length exactly $s_2 - 1$. □

Lemma 3.2. *Let* G *be an* (s_1, ℓ_P, s_2)*-barbel with* s_1 *and* s_2 *are odd and* $s1 \leq s_2$*. If* $\operatorname{diam}(G) > s_2 - 1$*, then* $\exp(G) = \operatorname{diam}(G)$.

Proof. It is clear from the definition of diameter of a graph and exponent of a graph that $\exp(G) \geq \operatorname{diam}(G)$. We need t show that for each pair of vertices u and v, there is a uv-walk of length $\operatorname{diam}(G)$.

Case 1 Both vertices $u, v \in C_1$ **or** $u, v \in C_2$**.** Assume that $u, v \in C_1$. By Corollary 2.2 the $\exp(C_1) = s_1 - 1$, hence for each pair of vertices u and v in C_1 there is a uv-walk of length k for each $k \geq s_1 - 1$. Therefore for each pair of vertices u and v in C_1 there is a uv-walk of length $\operatorname{diam}(G)$. Similar argument works for the case where $u, v \in C_2$.

Case 2 The vertex $u \in C_1$ **and the vertex** $v \in C_2$**.** We use the notation as in Case 2 of the proof of Lemma 3.1. Let p_{uv} be the shortest uv-path in G. If $\ell(p_{uv}) \equiv \operatorname{diam}(G) \bmod 2$, the we are done. Hence assume that $\ell(p_{uv}) \not\equiv \operatorname{diam}(G) \bmod 2$. This implies $\ell(p_{ux}) \neq \ell(p_{vy})$. If $\ell(p_{ux}) \leq \ell(p_{vy})$, then the path p'_{uv} consisting of the path p'_{uv_1}, the path P, and the path p_{v_2v} has length $\ell(p'_{uv}) = \frac{1}{2}(s_1 + s_2) + \ell_P + \ell(p_{ux}) - \ell(p_{vy}) \leq \frac{1}{2}(s_1 + s_2) + \ell_P - 1 = \operatorname{diam}(G)$. Notice that $\ell(p'_{uv}) \equiv \operatorname{diam}(G) \bmod 2$, hence p'_{uv} can be extended to a uv-walk of length $\operatorname{diam}(G)$. If $\ell(p_{ux}) > \ell(p_{vy})$, then the path p'_{uv} consisting the path p_{uv_1}, the path P, and the path p'_{v_2v} has length $\ell(p'_{uv}) = \frac{1}{2}(s_1 + s_2) + \ell_P + \ell(p_{vy}) - \ell(p_{ux}) \leq \operatorname{diam}(G)$. Since $\ell(p'_{uv}) \equiv \operatorname{diam}(G) \bmod 2$, hence p'_{uv} can be extended to a uv-walk of length $\operatorname{diam}(G)$.

Case 3 Both vertices $u, v \in P$**.** Let p_{uv} be the shortest uv-path and assume that $\ell(p_{uv}) \not\equiv \operatorname{diam}(G) \bmod 2$. Consider the walk w_{uv} consisting of the path p_{uv_1}, the cycle C_1 and the path p_{v_1v}, and the walk w'_{uv} consisting of the path p_{uv_2}, the cycle C_2, and the path p_{v_2v}. Then $\ell(w_{uv}) \equiv \operatorname{diam}(G) \bmod 2$ and $\ell(w'_{uv}) \equiv \operatorname{diam}(G) \bmod 2$.

Define $\ell'(w_{uv}) = \min\{\ell(w_{uv}), \ell(w'_{uv})\}$. Since $\ell(w_{uv}) + \ell(w'_{uv}) = s_1 + s_2 + 2\ell_P$, $\ell'(w_{uv}) \leq \frac{1}{2}(s_1 + s_2) + \ell_P$. Notice that $\ell'(w_{uv}) \equiv \mathrm{diam}(G) \bmod 2$. This implies $\ell'(w_{uv}) \leq \mathrm{diam}(G)$.

Case 4 The Vertex $u \in C_1$ or $u \in C_2$ and the vertex $v \in P$. Assume $u \in C_1$, $v \in P$ and the path p_{uv}, consisting of the path p_{uv_1} and the path $p_{v_1 v}$, has the property that $\ell(p_{uv}) \not\equiv \mathrm{diam}(G) \bmod 2$. Define the path p'_{uv} to be the path consisting of the path p'_{uv_1} and the path $p_{v_1 v}$. Then $\ell(p'_{uv}) \equiv \mathrm{diam}(G) \bmod 2$. Notice that

$$\ell(p'_{uv}) \leq s_1 + \ell_P - 1 < \frac{1}{2}(s_1 + s_2) + \ell_P - 1 = \mathrm{diam}(G).$$

Hence we can extend p'_{uv} to a uv-walk of length $\mathrm{diam}(G)$.

Now assume that $u \in C_2$, $v \in P$ and the shortest uv-path p_{uv} has the property that $\ell(p_{uv}) \not\equiv \mathrm{diam}(G) \bmod 2$. Consider the path p'_{uv} consisting of the path p'_{uv_2} and the path $p_{v_2 v}$. Then $\ell(p'_{uv}) \equiv \mathrm{diam}(G) \bmod 2$. If $\ell(p'_{uv}) \leq \mathrm{diam}(G) \bmod 2$, then we are done. So we assume $\ell(p'_{uv}) > \mathrm{diam}(G)$. Now consider the walk w_{uv} consisting of the path p_{uv_2}, the path P, the cycle C_1 and the path $p_{v_1 v}$. Then

$$\begin{aligned} \ell(w_{uv}) &= s_1 + 2\ell(p_{v_1 v}) + \ell(p_{v2v}) + \ell(p_{v_2 u}) \\ &\equiv \mathrm{diam}(G) \bmod 2 \end{aligned}$$

Since $\ell(w_{uv}) + \ell(p'_{uv}) = s_1 + s_2 + \ell_P$ and $\ell(p'_{uv}) > \mathrm{diam}(G)$, we have $\ell(w_{uv}) < \frac{1}{2}(s_1 + s_2) + \ell_P + 1$. On the other hand we have $\ell(w_{uv}) \equiv \mathrm{diam}(G) \bmod 2$, hence $\ell(w_{uv}) \leq \mathrm{diam}(G)$. We can now extend the walk w_{uv} to a uv-walk of length $\mathrm{diam}(G)$.

We conclude that for each pair of vertices u and v in G, there is a uv-walk of length exactly $\mathrm{diam}(G)$. Hence $\exp(G) \leq \mathrm{diam}(G)$. □

As a direct consequence of Lemma 3.1 and Lemma 3.2 we have the following result that characterizes the odd and even primitive (s_1, ℓ_P, s_2)-barbels.

Theorem 3.1. *Let G be a primitive (s_1, ℓ_P, s_2)-barbel with s_1 and s_2 are odd and $s_1 \leq s_2$. Then*

$$\exp(G) = \begin{cases} \textit{even}, & \textit{if } \mathrm{diam}(G) \leq s_2 - 1 \\ \textit{even}, & \textit{if } \mathrm{diam}(G) \textit{ is even and } \mathrm{diam}(G) > s_2 - 1 \\ \textit{odd}, & \textit{if } \mathrm{diam}(G) \textit{ is odd and } \mathrm{diam}(G) > s_2 - 1. \end{cases}$$

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