A FOURTH-ORDER FINITE DIFFERENCE SOLVER FOR POISSON'S EQUATION VIA THE HALF-SWEEP ARITHMETIC MEAN (HSAM) METHOD

J. Sulaiman[1], M. Othman[2], & M.K. Hasan[3]

[1] School of Science and Technology, Universiti Malaysia Sabah, Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia.

[2] Faculty of Computer Science and Information Technology, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.

[3] Faculty of Information Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor, Malaysia.

[1]jumat@ums.edu.my; [2]mothman@fsktm.upm.edu.my; [3]khatim@ftsm.ukm.my

Abstract

The main purpose of this paper is to validate the efficiency and accuracy of the Half-Sweep Arithmetic Mean (HSAM) method using the fourth-order finite difference approximation equation to solve one-dimensional Poisson's equation. In this paper, we formulate the Full-Sweep and Half-Sweep Arithmetic Mean methods, namely FSAM and HSAM respectively. Finally some computational experiments were conducted to prove that the fourth-order solver using the HSAM methods is superior to the FSAM method.

Keywords: Half-Sweep Iterative, Arithmetic Mean Algorithm, Fourth-Order Finite Difference, Poisson's equation

1. Introduction

Recently many researchers have formulated high-order approximation equations obtained by discretization of existent differential equations in mathematical models. For instance, formulation on the fourth-order approximation equations have discussed by Spotz (1995) and Gupta et al. (1997a-b) in order to get more accurate approximate solutions. Each approximation equation will lead a system of linear equations, where the character and complexity of its coefficient matrix depends on the order of the equation. Apart from these discretizations, solving systems of linear equations using the iterated approach is more interesting to be studied. Therefore, in this paper the Arithmetic Mean (AM) method (Ruggiero & Galligani 1990) will be considered to get approximate solutions for any system of linear equations. Actually this method is categorized as one of two-step iterative methods. Other two-step iterative methods can be considered such as the Alternating Group Explicit (AGE) (Evans & Sahimi, 1988), the Iterative Alternating Decomposition Explicit (IADE) (Sahimi, Ahmad & Bakar, 1992) and the Reduced Iterative Alternating Decomposition Explicit (RIADE) (Sahimi & Khatim, 2001) methods.

However, next section in this paper, we show on the combination of the full-sweep and half-sweep iterative into the Arithmetic Mean (HSAM) method and then called the Full-Sweep Arithmetic Mean (FSAM) and Half-Sweep Arithmetic Mean (HSAM) methods respectively, see Sulaiman et al. (2004). To validate the accuracy of the fourth-order solver and the efficiency of the HSAM method, let us consider one-dimensional Poisson's equation defined as

$$-\frac{d^2U}{dx^2} = f(x), a_0 \le x \le b_0 \tag{1}$$

subject to the boundary conditions

$$U(a_0) = \beta_0, \quad U(b_0) = \beta_1$$

and α is a variable, which depends on the type of metals and f is a continuous function.

Before further discussions on formulation of the fourth-order full-sweep and half-sweep finite difference approximation equations for problem (1), we shall restrict our discussion onto uniform node points only. Let assume the solution domain (1) can be uniformly divided into $m = 2^p, p \ge 2$ subinterval, which its distance, Δx defined as

$$\Delta x = \frac{(b_0 - a_0)}{m} = h, n = m - 1. \tag{2}$$

2. The Fourth-Order Half-Sweep Finite Difference Approximation

Referring in Fig. 1, the finite grid networks show the distribution of uniform node points to be considered in implementing the half-sweep and full-sweep iterative methods. The applications of both methods into the Arithmetic Mean method will compute approximate solutions onto node points of type ◉ only until the iterative convergence is satisfied. While solutions of other remain points are computed directly, see Abdullah (1991), Ibrahim and Abdullah (1995), Yousif and Evans (1995), Abdullah and Ali (1996).

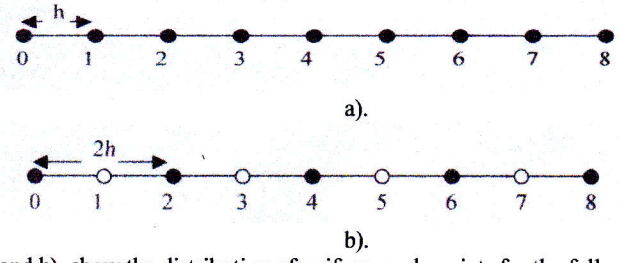

Fig. 1 a). and b). show the distribution of uniform node points for the full- and half-sweep cases respectively.

Using the finite difference approach, there are many approximation equations can be derived for problem (1). For instance, the second-order finite difference approximation equation can be expressed as

$$-U_{i-1} + 2U_i - U_{i+1} = h^2 f_i \tag{3}$$

Furthermore, the computational molecule for equation (3) is shown in Fig. 2 and the corresponding system of linear equations can be stated as

$$\underset{\sim}{A}\underset{\sim}{U} = \underset{\sim}{b} \tag{4}$$

where,

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(nxn)},$$

$$\underset{\sim}{U} = \begin{bmatrix} U_1 & U_2 & U_3 & \cdots & U_n \end{bmatrix}^T$$

$$\underset{\sim}{f} = \begin{bmatrix} h^2 f_1 + U_0 & h^2 f_1 & h^2 f_2 & \cdots & h^2 f_n + U_{n+1} \end{bmatrix}^T$$

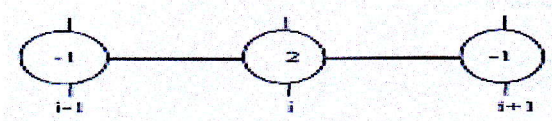

Fig. 2 Computational molecule of the second-order finite difference scheme for problem (1)

Using the same step to get approximation equation (3), it can be shown that the fourth-order full-sweep and half-sweep finite difference approximation equations generally can be expressed as

$$U_{i-p} - 16U_{i-p} + 30U_i - 16U_{i+p} + U_{i+p} = 12(ph)^2 f_i \tag{5}$$

The value of p, which corresponds to 1 and 2, represents the full-sweep and half-sweep cases respectively. In addition, Fig. 3 shows the computational molecule for both fourth-order finite-difference schemes. Based on Fig. 1 and 3, we can not compute approximate values for node points, $i = p$ and $i = m - p$ by using equation (5), because some node points are out of the solution domain (1), see Fig. 4. To overcome this problem, both points will be evaluated by equation (3).

Therefore, combination between equations (3) and (5) can be easily shown in a matrix form generally stated as

$$\underset{\sim}{A}\underset{\sim}{U} = \underset{\sim}{b} \tag{6}$$

where,

$$A=\begin{bmatrix} 2 & -1 & & & & & \\ -16 & 30 & -16 & 1 & & & \\ 1 & -16 & 30 & -16 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -16 & 30 & -16 & 1 \\ & & & 1 & -16 & 30 & -16 \\ & & & & & -1 & 2 \end{bmatrix}_{\left(\left(\frac{m}{p}\right)-1\right)x\left(\left(\frac{m}{p}\right)-1\right)},$$

$$\underset{\sim}{U}=\begin{bmatrix} U_{1p} & U_{2p} & U_{3p} & \cdots & U_{m-p} \end{bmatrix}^T,$$

$$\underset{\sim}{f}=\begin{bmatrix} h^2 f_{1p}+U_0 & 12h^2 f_{2p}+U_0 & 12h^2 f_{3p} & \cdots & h^2 f_{m-p}+U_m \end{bmatrix}^T.$$

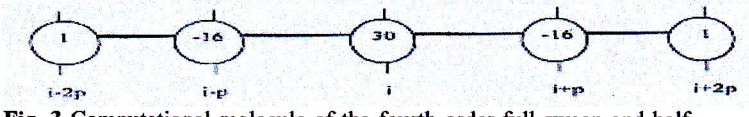

Fig. 3 Computational molecule of the fourth-order full-sweep and half-sweep finite difference scheme for problem (1).

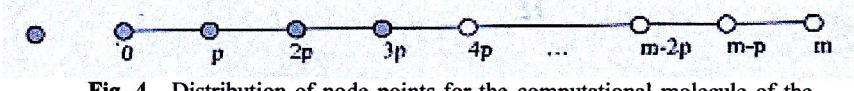

Fig. 4 Distribution of node points for the computational molecule of the fourth-order full-sweep and half-sweep schemes at $i = p$.

3. The Half-Sweep Arithmetic Mean Method

As explained in previous section, the AM method is categorized as one of two-step iterative methods. It means that the iterative process for this method involves two levels of virtual time such as $\underset{\sim}{U}^{(1)}$ and $\underset{\sim}{U}^{(2)}$. In this paper, we show on how to derive the FSAM or HSAM method. To facilitate on formulation of HSAM and FSAM methods onto system of equations (6), let us rewrite the coefficient matrix, A in a general form as

$$A\underset{\sim}{U}=\underset{\sim}{f} \tag{7}$$

where,

$$A=\begin{bmatrix} a_{1p} & b_{1p} & d_{1p} & & & & \\ c_{2p} & a_{2p} & b_{2p} & d_{2p} & & & \\ \rho_{3p} & c_{3p} & a_{3p} & b_{3p} & d_{3p} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \rho_{m-3p} & c_{m-3p} & a_{m-3p} & b_{m-3p} & d_{m-3p} \\ & & & \rho_{m-2p} & c_{m-2p} & a_{m-2p} & b_{m-2p} \\ & & & & \rho_{m-p} & c_{m-p} & a_{m-p} \end{bmatrix}_{\left(\left(\left(\frac{m}{p}\right)-1\right)x\left(\left(\frac{m}{p}\right)-1\right)\right)}$$

Thereafter let assume the matrix, A in equation (7) needs to be decomposed into

$$A=L+D+T \tag{8}$$

where L, D and T are lower triangular, diagonal and upper triangular matrices respectively. The general scheme for both AM methods is given by (Ruggiero & Galligani 1990; Sulaiman et al. 2004)

$$\left.\begin{aligned} (D+rL)\underset{\sim}{U}^{(1)} &= ((1-r)D-rT)\underset{\sim}{U}^{(k)}+r\underset{\sim}{f} \\ (D+rT)\underset{\sim}{U}^{(2)} &= ((1-r)D-rL)\underset{\sim}{U}^{(k)}+r\underset{\sim}{f} \\ \underset{\sim}{U}^{(k+1)} &= \frac{1}{2}\left(\underset{\sim}{U}^{(1)}+\underset{\sim}{U}^{(2)}\right) \end{aligned}\right\} \tag{9}$$

where r and $\underset{\sim}{U}^{(k)}$ represent as an acceleration parameter and an unknown vector at the k^{th} iteration respectively. Practically the value of r will be obtained by implementing some computer programs and then choose one optimal value of r, where its number of iterations is the smallest. Getting values of matrices L, D and T as stated in (8), the general algorithm for FSAM and HSAM schemes in (9) may be described in Algorithm 1.

Algorithm 1. FSAM and HSAM schemes

i) at level (1)

a. For $i=1p,2p,3p,\cdots,m-p$, calculate

$$U_i^{(1)} \leftarrow \begin{cases} \left(a_i(1-r)U_i^{(k)}-b_irU_{i+1p}^{(k)}-d_irU_{i+2p}^{(k)}+rf_i\right)/a_i & ,i=1p \\ \left(-c_irU_{i-1p}^{(1)}+a_i(1-r)U_i^{(k)}-b_irU_{i+1p}^{(k)}-d_irU_{i+2p}^{(k)}+rf_i\right)/a_i & ,i=2p \\ \left(-\rho_irU_{i-2p}^{(1)}-c_irU_{i-1p}^{(1)}+a_i(1-r)U_i^{(k)}-b_irU_{i+1p}^{(k)}+rf_i\right)/a_i & ,i=m-2p \\ \left(-\rho_irU_{i-2p}^{(1)}-c_irU_{i-1p}^{(1)}+a_i(1-r)U_i^{(k)}+rf_i\right)/a_i & ,i=m-p \\ \left(-\rho_irU_{i-2p}^{(1)}-c_irU_{i-1p}^{(1)}+a_i(1-r)U_i^{(k)}-b_irU_{i+1p}^{(k)}-d_irU_{i+2p}^{(k)}+rf_i\right)/a_i & ,Others \end{cases}$$

ii) at level (2)

b. For $i = m-p, m-2p, \cdots, 2p, 1p$, calculate

$$U_i^{(2)} \leftarrow \begin{cases} \left(-d_i r U_{i+2p}^{(2)} - b_i r U_{i+1p}^{(2)} + a_i(1-r)U_i^{(k)} + rf_i\right)/a_i & ,i = 1p \\ \left(-d_i r U_{i+2p}^{(2)} - b_i r U_{i+1p}^{(2)} + a_i(1-r)U_i^{(k)} - c_i r U_{i-1p}^{(k)} + rf_i\right)/a_i & ,i = 2p \\ \left(-b_i r U_{i+1p}^{(2)} + a_i(1-r)U_i^{(k)} - c_i r U_{i-1p}^{(k)} - \rho_i r U_{i-2p}^{(k)} + rf_i\right)/a_i & ,i = m-2p \\ \left(a_i(1-r)U_i^{(k)} - c_i r U_{i-1p}^{(k)} - \rho_i r U_{i-2p}^{(k)} + rf_i\right)/a_i & ,i = m-p \\ \left(-d_i r U_{i+2p}^{(2)} - b_i r U_{i+1p}^{(2)} + a_i(1-r)U_i^{(k)} - c_i r U_{i-1p}^{(k)} - \rho_i r U_{i-2p}^{(k)} + rf_i\right)/a_i & ,Others \end{cases}$$

c. For $i = 1p, 2p, 3p, \cdots, m-p$, calculate

$$U_i^{(k+1)} \leftarrow \frac{1}{2}\left(U_i^{(1)} + U_i^{(2)}\right)$$

The FSAM and HSAM algorithms are explicitly performed by using all equations at level (1) and level (2) alternatively until the specified convergence criterion is satisfied. Then the Full-Sweep Gauss-Seidel (FGS) method acts as the control of comparison of numerical results.

4. Computational Experiments

To show the efficiency of the HSAM scheme using the fourth-order finite difference approximation equation (5) especially in term of the number of iterations, execution time and maximum absolute error, we conducted numerical experiments to solve the one-dimensional Poisson's equation as follows

$$-\frac{d^2U}{dx^2} = 9\sin(3x), \qquad x \in [0,1] \tag{10}$$

Then boundary conditions and the exact solution of the problem (10) were defined by

$$U(x) = \sin(3x), \qquad 0 \le x \le 1. \tag{11}$$

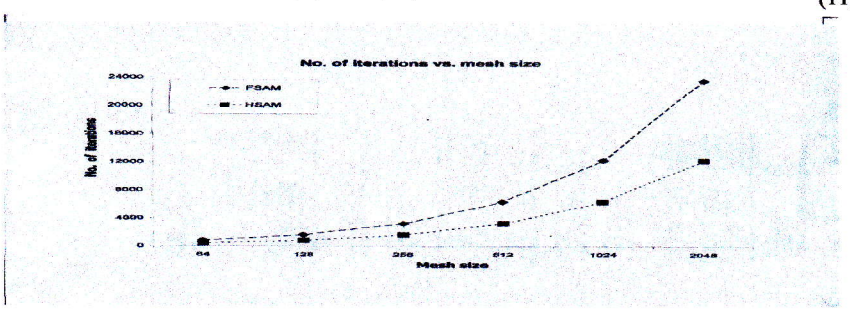

Fig. 5 Number of iterations versus mesh size of the FSAM and HSAM methods.

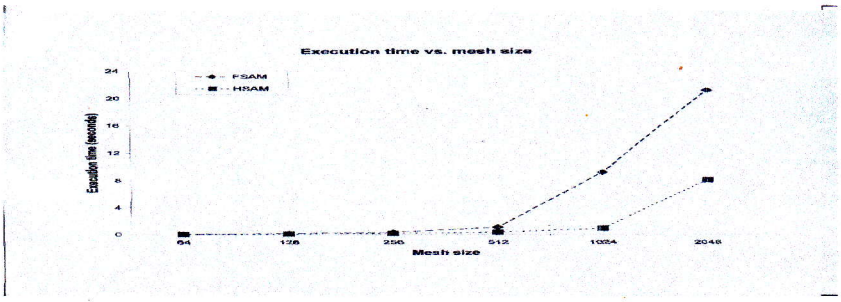

Fig. 6 The execution time (seconds) versus mesh size of the FSAM and HSAM methods.

All results of numerical experiments, obtained from implementation of the GS, FSAM and HSAM methods, have been recorded in Table 1. In the implementation mentioned above, the convergence test considered the tolerance error $\varepsilon = 10^{-10}$. Fig. 5 and 6 show number of iterations and the execution time versus mesh size respectively.

5. Conclusion

From the observation in Table 1, the finding in Fig. 5 and 6 shows that a number of iterations and the execution time for the HSAM have declined by 48.05 – 50.90% and 50.00 – 90.24% respectively compared with the FSAM method. Overall, the numerical results shows that the HSAM method is superior to the FSAM method in terms of a number of iterations and the execution time. This is attributed to the computational complexity of the HSAM method, which is nearly 50% less than the FSAM method. In term of the accuracy of approximate solutions, the HSAM method with the fourth-order finite difference approximation equation has got its maximum absolute error more accurate than the FGS method with the second-order standard finite difference approximation equation.

Table 1: Comparison of a number of iterations, the execution time (seconds) and maximum errors for the iterative methods.

No. of Iterations						
Methods	Mesh size					
	64	128	256	512	1024	2048
FGS (2^{nd} Order)	7063	25950	94592	341534	1218827	4286118
FSAM (4^{th} Order)	837	1662	3258	6334	12245	23572
HSAM (4^{th} Order)	411	837	1662	3258	6334	12245
Execution time (Seconds)						
Methods	Mesh size					
	64	128	256	512	1024	2048
FGS (2^{nd} Order)	0.31	1.13	63.33	299.57	1205.42	3692.09
FSAM (4^{th} Order)	0.06	0.16	0.34	1.07	9.12	21.12
HSAM (4^{th} Order)	0.03	0.08	0.17	0.32	0.89	7.99

Maximum Absolute Errors						
Methods	Mesh size					
	64	128	256	512	1024	2048
FGS (2^{nd} Order)	1.697e-4	4.228e-5	9.948e-6	5.293e-8	9.961e-6	4.233e-5
FSAM (4^{th} Order)	1.057e-7	4.622e-9	1.790e-8	3.691e-8	7.460e-8	1.502e-8
HSAM (4^{th} Order)	2.280e-6	1.174e-7	4.742e-9	1.790e-8	3.691e-8	7.460e-8

References

1. A.R. Abdullah, "The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver", *International Journal Computer Mathematics.* **38**(1991) 61-70.
2. A.R. Abdullah, and N.H.M. Ali, 1996. "A comparative study of parallel strategies for the solution of elliptic pde's", *Parallel Algorithms and Applications.* **10**(1996) 93-103.
3. D.J Evans, and M.S. Sahimi, 1988. "The Alternating Group Explicit iterative method (AGE) to solve parabolic and hyperbolic partial differential equations", *Ann. Rev. Num. Fluid Mechanic and Heat Trans.* **2**(1988) 283-389.
4. M. M. Gupta, J. Kouatchou, and J. Zhang, "A Compact Multigrid solver for convection-diffusion equations", *Journal of Computational Physics.* **132**(1) (1997a) 123-129.
5. M. M. Gupta, J. Kouatchou, and J. Zhang, "Comparison of second and fourth order discretizations for multigrid Poisson solvers", *Journal of Computational Physics.* **132**(2) (1997b) 226-232.
6. A. Ibrahim, and A.R. Abdullah, "Solving the two-dimensional diffusion equation by the four point explicit decoupled group (EDG) iterative method", *International Journal Computer Mathematics.* **58**(1995) 253-256.
7. J. Sulaiman, M. Othman, and M.K. Hasan. "A new Half-Sweep Arithmetic Mean (HSAM) algorithm for two-point boundary value problems". Proceedings of the International Conference on Statistics and Mathematics and Its Application in the Development of Science and Technology. (2004) 169-173.
8. V. Ruggiero, and E. Galligani, 1990. "An iterative method for large sparse systems on a vector computer". *Computer Math. Applic.* **20**(1990) 25-28.
9. W.F. Spotz, "High-Order Compact Finite Difference Schemes for Computational Mechanics". (Ph.D Dissertation, The University of Texas at Austin, 1995).
10. M.S. Sahimi, A. Ahmad, and A.A Bakar, "The Iterative Alternating Decomposition Explicit (IADE) method to solve the heat conduction equation", *International Journal Computer Mathematic,* **47**(1993) 219-229.
11. M.S. Sahimi, and M. Khatim. "The Reduced Iterative Alternating Decomposition Explicit (RIADE) Method for the Diffusion Equation", *Pertanika J. Sci. & Technol.* **9**(1) (2001)13-20.
12. W.S. Yousif, and D.J. Evans, "Explicit De-coupled Group iterative methods and their implementations". *Parallel Algorithms and Applications.* **7**(1995) 53-71.