STABILITY IN TWO-STAGE STOCHASTIC INTEGER PROGRAMMING

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Abstract

There is a large number of different approaches for formulating and solving optimization problems under uncertainty. In applications, one is usually faces incomplete information on probabilty measure μ. Numerical attemps has been made mostly rely on approximating μ by "simpler" measures. This paper presents overview of stability of two-stage Stochastic Integer Programming model under perturbations of the integrating probability measure μ.

Keywords : *stochastic programming, complete integer recourse,*
Simple integer recourse, stability, probability measure.

1. Introduction

Problems of the stability of solutions in mathematical programming problems are considered from different viewpoints and investigated from various positions. In works devoted to the investigation of the stability of the solutions, the subject of observation is a conditional extremum which is taken either as a random point, or as a random value. Depending on that, quite different notions of stability are inserted. In a number of works the stability problem is investigated in stochastic and parametric plans and also the light of the error theory.

In the stochastic programming problem the decision of the stability problems become very important since in these problems the values of the parameters are random. Here, we consider two-stage integer stochastic programs where the sum of the first stage cost and the expectation with respect to P of the second stage cost has to be minimized. However in most applications P is not known exactly. Even if P is given, it might happen that the stochastic program cannot be solved due to technical limitations and one has to use a simpler approximating distribution that makes the problem solvable. Hence, one often has to deal with statistical models and approximations Q of P. Of course, since solutions and optimal values of the original problem containing the distribution P are of interest, it is necessary to have statements at hand about stability of stochastic programs with respect to perturbations of P.

There are a number of such stability results in literature, see [16] for a recent survey. Most of these results consist of Lipshitz continuity properties of solution and optimal values with respect to certain probability metrics $d(P,Q)$. Especially in the case that P is unknown, this may in the end not be completely satisfactory, because in this case the distance $d(P,Q)$ is also unknown. Hence the question arises whether it is possible to prove statistical statements about the accuracy of solution and optimal values. In particular, confidence sets may be of interest. Such statistical statements require the availability of some statistical estimates associated with P, e.g. independent identically distributed sample of P.

The latter called empirical estimates and they can be understood as the so-called empirical measure $Q = P_n$ with $n \in N$ denoting the sample size. Asymptotic properties of statistical estimators in stochastic programming have been studied intensively. For two-stage stochastic programs without integrality requirements much is known. For mixed integer two-stage stochastic programming programs, the situation is essentially different. In [18]conditions are given implying consistency, convergence rates and a law of the iterated logarithm for optimal values. For pure integer model, large deviation type results are derived. Much of this work is based on recent development of empirical process theory e.g. [19]

Modeling the uncertainty by random objects may lead to diverse stochastic programming problems. In this paper we concentrate on two-stage problem only.

In Stochastic Integer Programming, integrality constraints are imposed on the first stage and/or second stage variables. Problems become almost unbearable if integrality constraints are imposed on second stage decision variables.
This model will be explained by analysis of its mathematical formulation

$$\min\{f_1(x) + E_\mu \varphi(x,\omega) : x \in X\} \tag{1}$$

As can be seen, this formula (1) presents a decision with possible recourse actions. At the first stage a decision x is to be selected from a set X of feasible solutions. A decision x gives rise directly to costs $f_1(x), f_1 : X \mapsto R$ and in the second stage to costs that are involved if a recourse action on the decision is taken. Besides x this recourse action depends on the realization of initially unknown data represented by random object ω, which is an element of some probability space $((\Omega, A, \mu)$. For a realization ω the recourse cost is denoted by the function $\varphi : X \times \Omega \mapsto R$ of x and ω, and hence a random object itself. Since it becomes realized only after the decision x is made we include it in the total cost function by taking its expected value $E_\mu \varphi(x,\omega)$, which is a (non-random) function of x, depending on the probability measure μ.
Let

$$Q(x,\mu) = E_\mu \varphi(x,\omega) \tag{2}$$

The recourse action usually asks for an optimal decision that is formulated as :

$$\varphi(x,\omega) = \min\{f_2(y,\omega) : y \in Y(x,\omega)\} \tag{3}$$

Here, y represents the recourse action to be selected from the set of feasible recourse action $Y(x, \omega)$. As indicated by the notation this set is dependent on x and ω. The cost of the action y may also depend on ω and is therefore denoted as $f_2(y, \omega), f_2 : Y \times \Omega \mapsto R$.

2. Model of two-stage problem

A Stochastic Programming problem related to the two-stage recourse problem is the distribution model, where the objective is to determine the distribution of the random variable

$$z(\omega) = \min\{f_1(x) + \varphi(x, \omega) : x \in X\} \tag{4}$$

A standard form of two-stage integer stochastic programming problem with complete recourse formulation is as follows :

$$\min\{c^T x + Q'(x, \mu) : Ax \ge b, x \ge 0, x \in R^n\} \tag{5}$$

$$Q'(x, \mu) = E_\mu \varphi'(x, \omega) \tag{6}$$

$$\varphi'(x, \omega) = \min\{q(\omega)^T y : Wy \ge p(\omega) - T(\omega), y \ge 0, y \in Z^d\} \tag{7}$$

with $\{y \in R^d : Wy \ge z, y \ge 0\} \ne 0, \forall z \in R^s$.

In general $\varphi'(x, \omega)$ is discontinuous in x for every $\omega \in \Omega$. This implies that under a discrete probability distribution on ω, $\varphi'(x, \mu)$ is also discontinuous. If ω is continuously distributed, the situation is slightly better : $\varphi'(x, \mu)$ is continuous but still non convex [6]. Hence the solution methods that were successful for stochastic linear programming problem cannot be simply adapted to solve this stochastic integer programming problem with integer second stage variables.

Illustration presented below is extracted from [11] concerning SunDay Icecream Co.'s planning the location and capacity of its distribution centers to serve demands from its retailers in N cities, where the demands of the various retailers are quite uncertain but the probability distribution of the retailer demands is known. Let y_i be the 0-1 variable indicating whether a distribution center is to be located in city i ($ii = 1, \ldots, N$) and x_i denote the non-negative variable indicating the capacity of the distribution center located in city-i. The maximum capacity that can be located in the city-i is denoted by U_i. The cost parameters a_i and b_i denote the per-unit capacity cost and the fixed location cost for locating a distribution center in city-i, respectively. Let z_{ij} denote 0-1 variable indicating the assignment of retailer j to distribution center-i, and γ_{ij} denote the associated fixed assignment cost; d_j be the uncertain demand of the retailer in city-j ($j = (1, \ldots, N)$. All costs are assumed to be amortized to a weekly basis, and the demands and capacities are in tones of ice-cream.

Note that the locations, capacities and assignments of the distribution centers have to be decided prior to observing the actual demand, whereas the shortage penalty is incurred upon observing the actual demand volume in the future. Consequently, the q_i is an uncertain parameter, whose actual value depends on the uncertain future demand volume. Now suppose that SunDay has to make the location-capacity-assignment decisions prior to observing demand. However, instead of paying a penalty on shortages, it wishes to guarantee that the probability of a shortage at any of the distribution center is small ($\leq \varepsilon$). The problem formulation now becomes what is called a chance-constrained model :

$$\min \sum_{i=1}^{N}(a_i x_i + b_i y_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_{ij}z_{ij} \tag{10}$$

such that $\quad 0 \leq x_i \leq U_i y_i$

$$\sum_{i=1}^{N} z_{ij} = 1$$

$$\sum_{j=1}^{N} z_{ij} \leq y_i$$

$$y_i, z_{ij} \in \{0,1\} \qquad i, j = 1,...,N$$

and

$$P_r\left(\sum_{j=1}^{N} dz_{ij} \leq x_i, i = 1,...,N\right) \geq 1-\varepsilon \tag{11}$$

The last constraint in the above model bound the probability of a shortage from above. Such constraints are known as chance constraint or probabilistic constraints. Unless the underlying parameter distributions and/or the constraint structure are of very specific form, chance constraints are extremely difficult to deal with algorithmically.

Suppose the demand d is represented by the random object d, element of some probability space (Ω, A, μ). For a realization w the recourse cost is denoted by the function $Q^I(x,\mu) = E_\mu Q^I(x,d)$ At the first stage, a decision x is to be selected which gives rise directly to costs $c^T x$, and in the second stage to costs that are involved if a recourse action on the decision is taken. Besides x this recourse action depends on the realization of initially unknown data that are represented by the random object d element of some probability space (Ω, A, μ). Since it becomes realized only after the decisions x is made we include it in the total cost function by taking its expected value $E_\mu Q^I(x,d)$ which is a (non-random) function of x, depending on the probability measure μ. In stochastic integer programming, integrality constrains are imposed on the first stage and/or second stage variables. This introduces computational difficulties. In

general, $Q(x,d)$ is discontinuous in x for every $d \in \Omega$. This implies that under discrete probability distribution the situation is slightly better : $Q(x,d)$ is continuous but still non convex (see [8] and [5] section 3).

3. Approximation Methods

In stochastic (mixed) integer programming integrality constrains are imposed on (some of) the first stage and/or second stage variables. This causes computational difficulties and gives strong evidence that polynomial time algorithms for its solution do not exist (whereas such an algorithm has been devised for linear programming [20]). In case only the first stage variables are restricted to be integer, the nice properties of the second stage cost function are maintained, hence we obtain an integer convex programming problem.

There are three levels of difficulty in solving stochastic integer programs (5).
- Evaluating the second-stage cost for a fixed first stage decision and a particular realization of the uncertain parameters.
- Evaluating the expected second –stage cost for a fixed first-stage decision.
- Optimizing the expected second-stage cost.

Two main approaches have been pursued in the recent literature with respect to solution methods. One approach is to abolish optimality and settle for an approximation. This approach has been investigated by [8] chapter 1, 2, and 3.

With approximation methods, the goal is find a lower bound and upper bound of x as an approximation for the optimal first stage decision x^*. This is called confidence interval for optimal values.[13] has shown that limit theorems can be used to derive confidence interval for optimal value via a certain modification of the bootstrapping like method / sampling. Moreover they have done empirical approximations of two-stage stochastic mixed integer programs and derive central limit theorems for the objective and optimal values.

Probability theory provides the tools for establishing mathematical indications on the quality of the approximations. A common attitude in solving NP-hard combinatorial optimization problems is not to insist on optimality but dedicate research efforts to designing fast and high quality approximation methods. This approach was initiated by STOUGIE [6], DEMPSTER et. al. [7,8]

To analyze the quality of the approximations, the stochastic nature of the problems suggests a probabilistic error analysis.
The following illustration on results typical for this approach (worked by[5]) by an example concerning a hierarchical scheduling problem which is extracted from [8], where improvements of results in are obtained in [21].
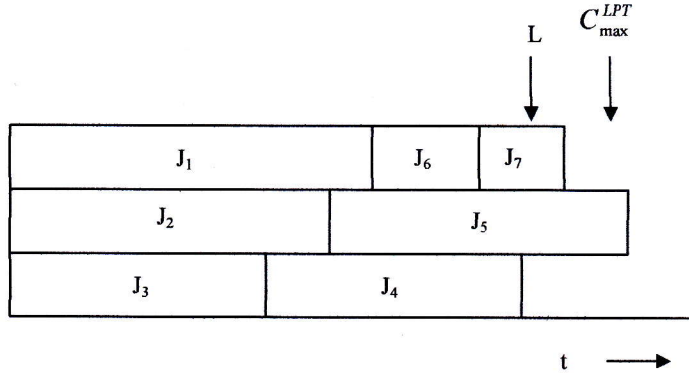
$$L \qquad C_{max}^{LPT}$$

J_1		J_6	J_7
J_2		J_5	
J_3		J_4	

$$t \longrightarrow$$

Figure 1 : Illustration of LPT rule

At the first stage or aggregate level we have to decide on the acquisition of a number x of identical and parallel machines, each of which costs an amount c. On the machines acquired a number n of jobs must be processes at the second stage or detailed level according to a schedule to be constructed such as to minimize the completion time C_{max} of the job completed last, the so-called makespan. Suppose each time unit costs 1. Each job j has a processing time $\pi_j(\omega)$, $j = 1,..., n$, the precise value of which is not known at the moment of the first stage decision. Assuming that $\pi_1(\omega), \pi_2(\omega),..., \pi_n(\omega)$ are independent and identically distributed (i.i.d.) random variables, where $!$ lives on some probability space (Ω, A, μ) with μ such that $E_\mu \pi_1^2(\omega) < \infty$. It is clear that the makespan is a function of the processing times and the number of machines available, which is reflected the notation $C_{max}(x, \pi(\omega))$ with $\pi(\omega) = (\pi_1(\omega),..., \pi_n(\omega))$.

The above lead to the stochastic programming problem

$$\min\{cx + E_\mu C_{max}^*(x, \pi(\omega)) : x \geq 0, x \in Z\} \tag{12}$$

Consider the longest processing time first (LPT) rule as a means for approximating it. This method orders the jobs on a list in decreasing processing time, and at each step the next jobs on a list is assigned to the earliest available machine (figure 1). The use of LPT is two fold. First it can serve as an approximation for the second stage problem. Secondly, it provides an upper bound on the optimal value that together with an obvious lower bound can be used to derive an estimate of the $C_{max}^*(x, \pi(\omega))$ that can becomes very accurate as the number of jobs becomes large. Let $C_{max}^{LPT}(x, \pi(\omega))$ denotes the solution value produced by LPT and $\pi_{max}(\omega) = \max_{j=1,...,n}\{\pi_j(\omega)\}$. Then for every $\omega \in \Omega$ we have :

$$\frac{1}{x}\sum_{j=1}^{n}\pi_j(\omega) \le C_{max}^*(x,\pi(\omega)) \le C_{max}^{LPT}(x,\pi(\omega) \le \frac{1}{x}\sum_{j=1}^{n}\pi_j(\omega) + \pi_{max}(\omega) \qquad (13)$$

[5] has solved the above problem and found $x^H \in \{ \left\lfloor \sqrt{\frac{n\theta}{c}} \right\rfloor, \left\lceil \sqrt{\frac{n\theta}{c}} \right\rceil \}$ as an approximation for the optimal first-stage decision x*.
Together with the LPT-rule for the second-stage scheduling problem this yields a solution value

$$z^H = cx^H + E_\mu C_{max}^{LPT}(x^H, \pi(\omega))$$

Writing the optimal value as $z^* = cx^* + E_\mu C_{max}^*(x^*, \pi(\omega))$, and using the bounds in (12) and taking expectations, it can be verified that

$$\frac{z^H}{z^*} \le 1 + \frac{E_\mu \pi_{max}(\omega)}{2\sqrt{n\theta c}} \qquad (14)$$

which establish the worst-case bound on the performance ratio of the approximation. Under the assumption made, $E_\mu \pi_1^2(\omega) < \infty$, we have that

$$\lim_{n\to\infty} \frac{E_\mu \pi_{max}(\omega)}{2\sqrt{n\theta c}} = 0 \qquad (15)$$

This approximation is said to be asymptotically optimal.

In the case the problem is under the assumption that all information is available in advanced (4), the approximation solution can be compared with the optimal solution. [5] has also worked on such a case and found that the relative error that can be attributed to imperfect information also tends to zero with probability one.

Given difficulty of solving stochastic programs, particularly stochastic integer programs, developing approximation algorithms for such problems is a promising direction for future research.

3. Stability of recourse problems

When formulating a stochastic programming model one assumes the underlying probability distribution to be given. In practical situations this is rarely the case. One often has to live with incomplete information and approximations. Furthermore, also under full information about the unde4rlying distributions one is led to approximations, since exact computation of expectations and probabilities typically arising in stochastic programming is beyond the present numerical capabilities for a large class of distributions. Since solutions and optimal values of the original problem containing the distribution P are of

interest, it is necessary to have statements at hand about stability of stochastic programs with respect to perturbations of P. These circumstances motivate a stability analysis for optimal values and optimal solutions to stochastic programs with respect to perturbations of the underlying probability distributions. There are many excellent recent papers on stability of stochastic integer programming problems e.g. {2,6,11]

In this section, what is investigated is the stability of optimal solutions and optimal values to the integer recourse problem $P(x, \lambda, \mu)$ when the probability distribution μ and weight parameter λ are subjected to perturbations. The stability of problem $P(x, \lambda, \mu)$ with r = 1 is examined with respect to variations both in μ and Q, see Kall [10] and Dupacova [11]. In the present paper we pursue this for two stage stochastic integer program with recourse. The scalar minimization problem $P(x, \lambda, \mu)$ is considered by Chen [12] and Cho [9]. In [6], the stability of $P(x, \lambda, \mu)$ with respect to variations in μ and λ in the sense of weak convergence by defining the bounded Lipschitz metric on distribution space.

There is some special work about confidence sets for solutions and optimal values of stochastic program. In [14], a stochastic programming with finite decision space is considered. Confidence sets for the solution set are desired by estimating the objective for each possible decision and selecting the presumably best decisions according to some statistical procedure. Statistical behaviour of the objective of general stochastic integer programming problems has been analyzed in [15]. Since the solution is within acceptable confidence interval, hence the question arises whether it is possible to prove statistical statements about the accuracy of solution and optimal values.

The stability of optimal solutions and optimal values to the linear recourse problem $P(x, \lambda, \mu)$ when the probability distribution μ and weight parameter λ are subjected to perturbations has been investigated by Cho et al.[9]. They derive quantitative continuity properties of the expectation functional which leads to qualitative and quantitative stability results for optimal values and optimal solutions with respect to perturbations of the underlying probability distributions and weight parameters in the objective function of the multiobjective programming problem.

Conclusion

To establish stability of two-stage stochastic integer program with complete recourse and to get the stability analysis on problem $P(x, \lambda, \mu)$, one needs to investigate a structural properties of a problem, mainly determined by the properties of the function Q^I given in (5),(6).If the stochastic programming problem is absolutely planning stable with the probability α, then since the optimal plan has with all corresponding realizations ω one and the same objective function's value, one may say that the problem is also functionally stable, at least, with the probability α.

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