NEW HOLDER - TYPE INEQUALITIES FOR THE TRACY-SINGH AND KHATRI-RAO PRODUCTS OF POSITIVE MATRICES

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Abstract

Recently, the authors established a number of inequalities involving Khatri-Rao product of two positive matrices. Here, in this paper, the results are established in three ways. First, we find new Holder-type inequalities for Tracy –Singh and Khatri-Rao products of positive semi-definite matrices. Second, the results are extended to provide estimates of sums of the Khatri-Rao and Tracy-Singh products of any finite number of positive semi-definite matrices. Three, the results lead to inequalities involving the Hadamard and Kronecker products, as a special case.

Keywords: Tracy-Singh Product, Khatri-Rao Product, Kronecker product, Hadamard Product , Positive semi-definite matrix.

1. Introduction

Consider matrices $A=[a_{ij}], C=[c_{ij}]\in M_{m,n}$ and $B=[b_{kl}]\in M_{p,q}$. Let A and B be partitioned as $A=[A_{ij}]$ and $B=[B_{kl}]$ $(1\le i\le t,\ 1\le j\le c)$, where A_{ij} is an $m_i\times n_j$ matrix and B_{kl} is a $p_k\times q_l$ matrix $(m=\sum_{i=1}^{t} m_i,\ \ n=\sum_{j=1}^{c} n_j,\ p=\sum_{i=1}^{t} p_i\ \ q=\sum_{j=1}^{c} q_j)$.

Let $A\otimes B$, $A\circ C$, $A\Theta B$ and $A*B$ be the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, respectively. The definitions of the mentioned four matrix products are given by (see e.g. [2])

$$A\otimes B=\left(a_{ij}B\right)_{ij}\ ;\ A\circ C=\left(a_{ij}c_{ij}\right)_{ij}\ ; \tag{1}$$

$$A*B=\left(A_{ij}\otimes B_{ij}\right)_{ij}\ ;\ A\Theta B=\left(A_{ij}\Theta B\right)_{ij}=\left(\left(A_{ij}\otimes B_{kl}\right)_{kl}\right)_{ij}. \tag{2}$$

Additionally, the Khatri-Rao product can be viewed as a generalized Hadamard product and the Tracy-Singh product as a generalized Kronecker product, i.e., for a non-partitioned matrix A and B, their $A\Theta B$ is $A\otimes B$ and $A*B$ is $A\circ B$. For any compatibly partitioned matrices A, B, C, D, we shall make frequent use the following properties of the Tracy-Singh product (see e.g. [1,2])

$$(A\Theta B)(C\Theta D) = (AC)\Theta(BD); \quad (3)$$

$$(A\Theta B)^* = A^*\Theta B^*. \quad (4)$$

An Hermitian matrix A is called positive semi-definite (Written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for any vector x. For Hermitian matrices A and B, the relations $A \geq B$ mean that $A - B \geq 0$ is a positive semi-definite. Given a positive semi-definite matrix A and $k \geq 1$ be a given integer, then there exists a unique positive semi-definite matrix B such that $A = B^k$, written as $B = A^{1/k}$. Let $A \in M_m$ be a positive definite. The *spectral decomposition* of A assures that there exists a unitary matrix U such that (see e.g. [6])

$$A = U^* D\, U = U^* diag(\lambda_i) U \quad , \quad U^* U = I_m . \quad (2)$$

Here, $D = diag(\lambda_i) = diag(\lambda_1, \cdots, \lambda_m)$ is the diagonal matrix with diagonal entries λ_i (λ_i are the positive egeinvalues of A). For any real number r , A^r is defined by

$$A^r = U^* D^r U = U^* diag(\lambda_i^r) U . \quad (3)$$

2. Some Lemmas

Denote H_n^+ be the set of all positive semi-definite $n \times n$ matrices. We present the following three Lemmas as basic results.

2.1. Lemma

Let $A_i \in M_{m(i),n(i)}$ $(1 \leq i \leq k, k \geq 2)$ are compatibly partitioned matrices ($m = \prod_{i=1}^{k} m(i), n = \prod_{i=1}^{k} n(i)$, $r = \sum_{j=1}^{t}\prod_{i=1}^{k} m_j(i)$, $s = \sum_{j=1}^{c}\prod_{i=1}^{k} n_j(i)$, $m(i) = \sum_{j=1}^{t} m_j(i)$, $n(i) = \sum_{j=1}^{c} n_j(i)$). Then (see e.g. [1]) there exists two real matrices Z_1 of order $m \times r$ and Z_2 of order $n \times s$ such that $Z_1^T Z = I_1$, $Z_2^T Z = I_2$ and

$$\prod_{i=1}^{k} * A_i = Z_1^T\left(\prod_{i=1}^{k} \Theta A_i\right) Z_2 . \quad (4)$$

Here, I_1 and I_2 are identity matrices of order $r \times r$ and $s \times s$, respectively.

2.2. Lemma

Let a_i and b_i ($1 \leq i \leq k$) be positive scalars. If $1 \leq p, q < \infty$ satisfy $(1/p) + (1/q) = 1$. Then the *scalar Holder inequality* is given by (see e.g., [7])

$$\sum_{i=1}^{k} a_i b_i \le \left(\sum_{i=1}^{k} a_i^p \right)^{1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q} . \tag{5}$$

2.3. Lemma

Let a_i and b_i ($1 \le i \le k$) be positive scalars. If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then

$$\sum_{i=1}^{k} a_i b_i \ge \left(\sum_{i=1}^{k} a_i^{-p} \right)^{-1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q} . \tag{6}$$

Proof: The condition $(1/q) - (1/p) = 1$ can be rewritten as $1/(p/q) + 1/(1/q) = 1$. Note that $1 \le (p/q) < \infty$ and $1 \le (1/q) < \infty$. By scalar Holder inequality in Lemma (2.2), we have

$$\sum_{i=1}^{k} b_i^q = \sum_{i=1}^{k} a_i^{-q} (a_i b_i)^q \le \left(\sum_{i=1}^{k} (a_i^{-q})^{p/q} \right)^{q/p} \left(\sum_{i=1}^{k} \{(a_i b_i)^q\}^{1/q} \right)^q$$
$$= \left(\sum_{i=1}^{k} a_i^{-p} \right)^{q/p} \left(\sum_{i=1}^{k} \{a_i b_i\} \right)^q .$$

Hence, $\sum_{i=1}^{k} a_i b_i \ge \left(\sum_{i=1}^{k} a_i^{-p} \right)^{-1/p} \left(\sum_{i=1}^{k} b_i^q \right)^{1/q}$. □

3. Main Results

3.1. Theorem

Let $A_i \in H_n^+$ be commutative partitioned matrices and $B_i \in H_m^+$ be commutative partitioned matrices ($1 \le i \le k$). If $1 \le p, q < \infty$ satisfy $(1/p) + (1/q) = 1$. Then

$$\sum_{i=1}^{k} A_i \Theta B_i \le \left(\sum_{i=1}^{k} A_i^p \right)^{1/p} \Theta \left(\sum_{i=1}^{k} B_i^q \right)^{1/q} \tag{7}$$

Proof: By assumption there exist a unitary matrix $U \in M_n$ and a unitary matrix $V \in M_m$ such that $A_i = U^* D_i U$ with $D_i = diag(d_{i1}, \cdots, d_{in})$ and $B_i = V^* T_i V$ with $T_i = diag(t_{i1}, \cdots, t_{im})$, where d_{ij}, t_{ij} are nonnegative real numbers for all i and j. It follows that

$$A_i \Theta B_i = (U^* D_i U) \Theta (V^* T_i U) = (U^* \Theta V^*)(D_i \Theta T_i)(U \Theta V)$$
$$= (U \Theta V)^* diag(d_{i1} t_{i1}, \cdots, d_{i1} t_{im}, \cdots, d_{in} t_{i1}, \cdots, d_{in} t_{im})(U \Theta V)$$

So, by using Lemma (2.2), we have

$$\sum_{i=1}^{k} A_i \Theta B_i = (U\Theta V)^* \, diag\left(\sum_{i=1}^{k} d_{i1}t_{i1},\cdots,\sum_{i=1}^{k} d_{i1}t_{im},\cdots,\sum_{i=1}^{k} d_{in}t_{i1},\cdots,\sum_{i=1}^{k} d_{in}t_{im}\right)(U\Theta V)$$

$$\leq (U\Theta V)^* \, diag\left[\left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}\right.$$
$$\left.\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}\left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q}\right](U\Theta V)$$

$$= (U\Theta V)^* \left\{ diag\left[\left(\sum_{i=1}^{k} d_{i1}^p\right)^{1/p},\cdots,\left(\sum_{i=1}^{k} d_{in}^p\right)^{1/p}\right]\right.$$
$$\left.\Theta diag\left[\left(\sum_{i=1}^{k} t_{i1}^q\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} t_{im}^q\right)^{1/q}\right]\right\}(U\Theta V)$$

$$= (U^*\Theta V^*)\left\{\left(\sum_{i=1}^{k} D_i^p\right)^{1/p}\Theta\left(\sum_{i=1}^{k} T_i^q\right)^{1/q}\right\}(U\Theta V)$$

$$= \left\{\left(U^*\left(\sum_{i=1}^{k} D_i^p\right)^{1/p} U\right)\Theta\left(V^*\left(\sum_{i=1}^{k} T_i^q\right)^{1/q} V\right)\right\}$$

$$= \left(\sum_{i=1}^{k} A_i^p\right)^{1/p}\Theta\left(\sum_{i=1}^{k} B_i^q\right)^{1/q}. \quad \square$$

3.2. Corollary

Let $A_i \in H_n^+$ be commutative partitioned matrices and $B_i \in H_m^+$ be commutative partitioned matrices $(1 \leq i \leq k)$. If $1 \leq p, q < \infty$ satisfy $(1/p) + (1/q) = 1$. Then

$$\sum_{i=1}^{k} A_i * B_i \leq \left(\sum_{i=1}^{k} A_i^p\right)^{1/p} * \left(\sum_{i=1}^{k} B_i^q\right)^{1/q}. \tag{8}$$

Proof: Follows immediately by applying Lemma (2.1) and Theorem (3.1). □

3.3. Corollary

Let $A_i^{(j)} \in H_{n^{(j)}}^+$ $(1 \leq i \leq k)$ be commutative partitioned $n^{(j)} \times n^{(j)}$

matrices, $(1 \leq j \leq r)$. Let $1 \leq \{p^{(j)}\}_{j=1}^r < \infty$ satisfy $\sum_{j=1}^{r}(1/p^{(j)}) = 1$. Then

$$\sum_{i=1}^{k}\left(\prod_{j=1}^{r}\Theta A_i^{(j)}\right) \leq \prod_{j=1}^{r}\Theta\left(\sum_{i=1}^{k}(A_i^{(j)})^{p^{(j)}}\right)^{1/p^{(j)}} \tag{9}$$

Proof: Using Theorem (3.1), the corollary follows by induction on k. □

3.4. Corollary

Let $A_i^{(j)} \in H^+_{n^{(j)}}$ $(1 \le i \le k)$ be commutative partitioned $n^{(j)} \times n^{(j)}$ matrices, $(1 \le j \le r)$. Let $1 \le \{p^{(j)}\}_{j=1}^{r} < \infty$ satisfy $\sum_{j=1}^{r}(1/p^{(j)}) = 1$. Then

$$\sum_{i=1}^{k}\left(\prod_{j=1}^{r} * A_i^{(j)}\right) \le \prod_{j=1}^{r} * \left(\sum_{i=1}^{k}(A_i^{(j)})^{p^{(j)}}\right)^{1/p^{(j)}} . \tag{10}$$

Proof: Using Corollary (3.3) and Lemma (2.1), the corollary follows by induction on k .□

We give an example using products of three matrices ($r = 3$). Let $A_i^{(j)}$ $(1 \le i \le k)$ be commutative positive partitioned $n \times n$ matrices, $(1 \le j \le 3)$. Let $1 \le \{p^{(j)}\}_{j=1}^{3} < \infty$ satisfy $(1/p^{(1)}) + .(1/p^{(2)}) + (1/p^{(3)}) = 1$. Then

(*i*) $\sum_{i=1}^{k} A_i^{(1)} \Theta A_i^{(2)} \Theta A_i^{(3)}$

$$\le \left(\sum_{i=1}^{k} A_i^{p^{(1)}}\right)^{1/p^{(1)}} \Theta \left(\sum_{i=1}^{k} A_i^{p^{(2)}}\right)^{1/p^{(2)}} \Theta \left(\sum_{i=1}^{k} A_i^{p^{(3)}}\right)^{1/p^{(3)}} . \tag{11}$$

(*ii*) $\sum_{i=1}^{k} A_i^{(1)} * A_i^{(2)} * A_i^{(3)}$

$$\le \left(\sum_{i=1}^{k} A_i^{p^{(1)}}\right)^{1/p^{(1)}} * \left(\sum_{i=1}^{k} A_i^{p^{(2)}}\right)^{1/p^{(2)}} * \left(\sum_{i=1}^{k} A_i^{p^{(3)}}\right)^{1/p^{(3)}} . \tag{12}$$

3.5. Theorem

Let $A_i \in H_n^+$ be commutative partitioned matrices and $B_i \in H_m^+$ be commutative partitioned matrices $(1 \le i \le k)$. If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then

$$\sum_{i=1}^{k} A_i \Theta B_i \le \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} \Theta \left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q} \tag{13}$$

Proof: By assumption there exist a unitary matrix $U \in M_n$ and a unitary matrix $V \in M_m$ such that $A_i = U^* D_i U$ with $D_i = diag(d_{i1}, \cdots, d_{in})$ and $B_i = V^* T_i V$ with $T_i = diag(t_{i1}, \cdots, t_{im})$, where d_{ij}, t_{ij} are nonnegative real numbers for all i and j. It follows that

$$\begin{aligned} A_i \Theta B_i &= \left(U^* D_i U\right)\Theta\left(V^* T_i U\right) = \left(U^* \Theta V^*\right)\left(D_i \Theta T_i\right)\left(U \Theta V\right) \\ &= \left(U \Theta V\right)^* diag(d_{i1}t_{i1}, \cdots, d_{i1}t_{im}, \cdots, d_{in}t_{i1}, \cdots, d_{in}t_{im})\left(U \Theta V\right) \end{aligned}$$

So, by using Lemma (2.3), we have

$$\sum_{i=1}^{k} A_i \Theta B_i = (U\Theta V)^* diag\left(\sum_{i=1}^{k} d_{i1}t_{i1},\cdots,\sum_{i=1}^{k} d_{i1}t_{im},\cdots,\sum_{i=1}^{k} d_{in}t_{i1},\cdots,\sum_{i=1}^{k} d_{in}t_{im}\right)(U\Theta V)$$

$$\geq (U\Theta V)^* diag\left[\left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right.$$

$$\left.,\cdots,\left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right](U\Theta V)$$

$$= (U\Theta V)^* \left\{ diag\left[\left(\sum_{i=1}^{k} d_{i1}^{-p}\right)^{-1/p},\cdots,\left(\sum_{i=1}^{k} d_{in}^{-p}\right)^{-1/p}\right]\right.$$

$$\left.\Theta diag\left[\left(\sum_{i=1}^{k} t_{i1}^{q}\right)^{1/q},\cdots,\left(\sum_{i=1}^{k} t_{im}^{q}\right)^{1/q}\right]\right\}(U\Theta V)$$

$$= (U^*\Theta V^*)\left\{\left(\sum_{i=1}^{k} D_i^{-p}\right)^{-1/p}\Theta\left(\sum_{i=1}^{k} T_i^{q}\right)^{1/q}\right\}(U\Theta V)$$

$$= \left\{\left(U^*\left(\sum_{i=1}^{k} D_i^{-p}\right)^{-1/p} U\right)\Theta\left(V^*\left(\sum_{i=1}^{k} T_i^{q}\right)^{1/q} V\right)\right\}$$

$$= \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p}\Theta\left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q}. \square$$

3.6. Corollary

Let $A_i \in H_n^+$ be commutative partitioned matrices and $B_i \in H_m^+$ be commutative partitioned matrices $(1 \leq i \leq k)$. If $0 < p < \infty$ and $0 < q < 1$ satisfy $(1/q) - (1/p) = 1$. Then

$$\sum_{i=1}^{k} A_i * B_i \geq \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-1/p} * \left(\sum_{i=1}^{k} B_i^{q}\right)^{1/q} \tag{14}$$

Proof: Follows immediately by Lemma (2.1) and Theorem (3.5). □

3.7. Remark

The results obtained in section 3 are quite general. Now, as a special case, consider the matrices in section 3 are non-partitioned, we then have Holder type inequalities involving Kronecker and Hadamard products by replacing Θ by $\otimes$ and $*$ by $\circ$.

4. Conclusion

The problem may occur that we can't find Holder-type inequalities for usual product of positive matrices, but here, we can find new Holder-type inequalities for the Tracy-Singh, Khatri-Rao, Kronecker and Hadamard products of positive matrices which are very important for applications to establish new inequalities involving these products. Since its sometimes difficult to compute, for example, ranks, determinants, eigenvalues, norms of large matrices, its of great importance to provide estimates of sums of these products of any finite number of matrices by applying Holder-type inequalities of positive matrices.

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