ON EXPONENTS OF PRIMITIVE GRAPHS

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Abstract

A connected graph G is primitive provided there exists a positive integer k such that for each pair of vertices u and v in G there is a walk of length k that connects u and v. The smallest of such positive integers k is called the exponent of G and is denoted by $\exp(G)$. In this paper, we give a new bound on exponent of primitive graphs G in terms of the length of the smallest cycle of G. We show that the new bound is sharp and generalizes the bounds given by Shao and Liu et. al.

Keywords: primitive graphs; exponents.

1. Introduction

Let G be a finite graph on n vertices. We follow notation and terminologies of graphs in Brualdi and Ryser [1]. Particularly, a *walk* of length m from a vertex u to a vertex v is a sequence of edges of the form

$$\{u=v_0,v_1\},\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{m-1},v_m\}$$

or

$$v_0-v_1-v_2-\cdots-v_{m-1}-v_m\,.$$

A walk w connecting vertices u and v is abbreviated by a *uv-walk* or w_{uv} and its length is denoted by $\ell(w_{uv})$. A *uv-path* is a *uv*-walk with no repeated vertices except possibly $u=v$. A *uv*-walk is *open* provided that $u\neq v$, and is closed otherwise. A *cycle* is a closed path and a *loop* is a cycle of length 1. A graph G is *connected* provided that for each pair of vertices u and v in G there is a *uv*-walk connecting u and v. A *tree* is a connected graph which has no cycles. A connected graph G is *primitive* provided there is a positive integer k such that for each pair of vertices u and v there is a *uv*-walk of length k. The smallest of such positive integer k is called the *exponent* of the graph G and is denoted by $\exp(G)$. The exponent set of graphs on n vertices is the set of all possible exponents of graphs on n vertices.

The following proposition (see [1]) gives necessary and sufficient conditions for primitivity of a connected graph.

Proposition 1. *Let G be a connected graph. The graph G is primitive if and only if G has cycle of odd length.*

Research on exponents of strongly connected directed graph dated back in 1950 and was initiated by Wielandt (see [4, 5]). For a primitive directed graph D on n vertices Wielandt showed that the $\exp(D) \leq (n-1)^2 + 1$. Dulmage and Mendelsohn (see[]) give a bound of exponent of primitive directed graph in terms of the length of the smallest directed cycles. They show that for a primitive directed graph D on n vertices the $\exp(D) \leq n + s(n-2)$, where s is the length of the smallest cycle in D.

For a primitive graph on n vertices, Shao (see [3]) showed that the $\exp(G) \leq 2n-2$. Moreover Shao showed that the exponent set of graphs on n vertices is the set $\{1, 2, \ldots, 2n-2\} \backslash S_n$ where S_n is the set of odd integers in the closed interval $[n, 2n-2]$. For primitive loopless graphs G on n vertices, Liu et.al (see [2]) showed that the exponent set of such graphs is the set $\{2, 3, \ldots, 2n-4\} \backslash S_n$ where S_n is the set of odd integers in the closed interval $[n-2, 2n-5]$. Let G be a primitive graph on n vertices with the smallest odd cycle of length s. This paper gives new upper bound for $\exp(G)$ in terms of s. We show that this bound is sharp and generalizes the bounds given by Shao and Liu et.al.

2. Facts about primitive graphs

This section discusses several facts concerning primitive graphs. One feature of primitive graphs is given by the following lemma that says for each pair of vertices u and v there is a uv-walk of length even.

The following proposition guarantees that any uv-walk of length even can be extended to a longer uv-walk of any length even.

Lemma 2. *Let G be a connected graph and let w a uv-walk. Then w can be extended to a uv-walk of length* $t \geq \ell(w) + 2k$ *for every positive integer k.*

Lemma 3. *Let G be a primitive graph. Then every uv-path can be extended to a uv-walk of length even.*

Proof. Since G is a primitive graph, G has a cycle C of odd length. Since G is connected, for each vertex v in $G \backslash C$ and a vertex c in the cycle C there is a vc-path that connects v and the vertex c.

Let u and v be any two vertices in G. Suppose p_{uv} is any uv-path that connects u and v. If the length of p_{uv} is even, then p_{uv} is a uv-walk of length even and we are done. So we assume that the length of p_{uv} is odd. We consider three cases. They are the case when p_{uv} and C have one vertex in common, p_{uv} and C have a path in common, and p_{uv} and C have no vertex in common.

Case 1: The path p_{uv} and C have a vertex in common.
Suppose c is the vertex in common between the path p_{uv} and the cycle C. Hence the path p_{uv} can be decomposed into the path p_{uc} and the path p_{cv}. Consider the uv-walk w created by moving from u to v as follows. We start at u and follow the path p_{uc} from u to c, then

follow the cycle C back to c, and finally follow the path p_{cv} from c and end at v. Then w is a uv-walk of length even.

Case 2: The path p_{uv} and C have a path in common.
Suppose path p_{cx} is the path in common between the path p_{uv} and the cycle C. Then the path p_{uv} can be decomposed into the path p_{uc}, p_{cx} and p_{xv}. Let w be the walk that starts at u, follows the path p_{uc} to c, follows the cycle C back to c, follows the path p_{cx} to x, and finally follows the path p_{xv} to end at v. Then w is a uv-walk of length even.

Case 3: The path p_{uv} and C have no vertex in common.
Let p_{uc} be a path that connects the vertex u and a vertex c in the cycle C. The walk w that starts at u, follows the path p_{uc} to the vertex c in C, moves around the cycle C back to c, follows the path p_{cu} back to u, and finally follows the path p_{uv} to v is a uv-walk of length even.

Therefore, each uv-path can be extended to a uv-walk of length even. ∎

Similarly one can show the following fact.

Lemma 4 *Let G be a primitive graph. Then every uv-path can be extended to a uv-walk of odd length.*

3. New upper bound for exponents

In this section we give a new upper bound for exponent of a primitive graph in terms of the length of the smallest odd cycle. We then show that our bound is sharp and generalizes the bounds given by Shao and Liu et.al. Let G be a primitive graph and let C be the smallest cycle of odd length s in G. Let u be a vertex in G but not in C, and let p_{ux} be a path that connects u and a vertex x in C. Let $\ell(p_{uv})$ be the length of the path p_{ux} and define

$$\ell = \max_{u\in G\setminus C, x\in C} \{\ell(p_{ux})\}$$

Theorem 5 *Let G be a primitive graph with smallest cycle of length s. Then*

$$\exp(G) \le s + 2\ell - 1.$$

Proof. Let C be the cycle $c_1 - c_2 - \cdots - c_{s-1} - c_s - c_1$ of length s. For each pair of vertices u and v in G, we show there is a uv-walk of length $s+2\ell-1$. We note that since s is odd, then $s+2\ell-1$ is even. Let p_{uv} be a uv-path that connects u and v. It is not hard to show that for each pair of vertices u and v there is a uv-path such that $\ell(p_{uv}) \le s+2\ell-1$. If p_{uv} is of even length, Lemma 2 guarantees that the path p_{uv} can be extended to a uv-walk of length $s+2\ell-1$. So we assume that p_{uv} is of odd length. Lemma 3 guarantees that the path p_{uv} can be extended to a uv-walk w of length even. Next we show that we can choose the uv-walk w such that the length of w is exactly $s+2\ell-1$. We consider two cases: the case when p_{uv} and C have vertices in common and the case when p_{uv} and C have no vertices in common.

Case 1. The path p_{uv} and the cycle C have vertices in common.
We claim that there is a uv-walk w such that $\ell(w)$ is even and $\ell(w) \leq s+2\ell-1$. Suppose the path p_{uv} and the cycle C have exactly one vertex in common c. Then the walk w that starts at u, moves to c along the path p_{uc}, moves along the cycle C back to c, and finally moves to v along the path p_{cv} is a uv-walk of length even. Since p_{uv} is of odd length, then $\ell(p_{uv}) = \ell(p_{uc}) + \ell(p_{cv}) \leq 2\ell - 1$. This implies $\ell(w) \leq s+2\ell-1$. Proposition 2 implies there is a uv-walk of length exactly $s+2\ell-1$.

Now assume that the path p_{uv} and the cycle C have more than one vertex in common. Without loss of generality let the path $c_i - c_{i+1} - \cdots - c_{i+k+1}$ be the path of length k that lies in both p_{uv} and C. Consider the path p' where p' is the path

$$c_i - c_{i-1} - \cdots - c_1 - c_s - c_{s-1} - \cdots - c_{i+k+1}$$

The path p started at vertex u, moves to vertex c_i along the path p_{u,c_i}, then moves to vertex c_{i+k+1} along the path p', and finally moves to vertex v along the path $p_{c_{i+k+1},v}$ is a uv-path of length even. More over $\ell(p) \leq s+2\ell-1$. Lemma 2 implies that we can find a uv-walk of length exactly $s+2\ell-1$.

Case 2. The path p_{uv} and the cycle C have no vertex in common.
In this case all uv-path in G have no vertex in common with the cycle C. This implies there is a rooted subtree T of G rooted at c in C such that both vertices u and v are in T. We consider two cases. They are the case when v is in the uc-path from u to c and the case when v is not in the uc-path. Suppose v is in the uc-path. Then the walk w that starts at u, moves to v along the path p_{uv}, moves to c along the path p_{vc}, moves around the cycle C and back to c, and finally moves to v along the path p_{vc} is a uv-walk of even length. Since $\ell(p_{uv}) \leq \ell$, then $\ell(w) \leq s+2\ell-1$. Hence Lemma 2 implies that there is a uv-walk of length exactly $s+2\ell-1$.

Now assume v does not lie in the uc-path. Since the path p_{uv} is of odd length, the length of p_{uc} and p_{vc} are not the same. Without loss of generality we may assume that the path p_{uc} is shorter than the path p_{vc}, that is $\ell(p_{uc}) < \ell(p_{vc})$. Notices that the walk w that starts at u, moves along the path p_{uc} to c, moves around the cycle C and back to c, and finally moves along the path p_{cv} and ends at v, is a uv-walk of length even. Moreover, $\ell(w) \leq s+2\ell-1$ and hence Lemma 2 guarantess that we can find a uv-walk of length exactly $s+2\ell-1$.

Now we can conclude that for each pair of vertices u and v in G, there is a uv-walk of length exactly $s+2\ell-1$. Hence the $\exp(G) \leq s+2\ell-1$. ■

Notice that the bound given in Theorem 5 implies that in order to have a primitive graph with large exponent, then the graph should have small value of s and large value of ℓ. As a direct consequence of Theorem 5 we have the following corollary.

Corollary 6 *Let G be a primitive graph on n vertices. If G has a loop, then the* $\exp(G) \leq 2n-2$. *Otherwise, the* $\exp(G) \leq 2n-4$.

Proof. If G has a loop, then $s = 1$. This implies $l \leq n - 1$. Theorem 5 implies $\exp(G) \leq 2n - 2$. If G has no loops, then $s \geq 3$ and $l \leq n - 3$. Theorem 5 implies $\exp(G) \leq 2n - 4$. ■

Let G be a connected graph on n vertices. Let C $c_1 - c_2 - \cdots - c_{s-1} - c_s - c_1$ be a cycle of length s in G. Let P be the path $v_s - v_{s+1} - \cdots - v_n$ of length $n - s$. A (v_s, v_n)-*lollipop* is a connected graph consisting of a cycle C of length s and a path P of length $n - s$. For example, Figure 1 gives a (v_6,v_n)-lollipop.

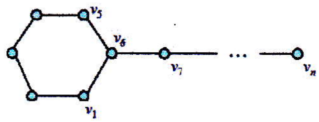

Fig. 1. (v_6,v_n)-*lollipop*

Shao shows that the graph G that attains the bound $\exp(G) \leq 2n - 2$ is the (v_1,v_n)-lollipop. While Liu et.al. show that the graph that attains the bound $\exp(G) \leq 2n - 4$ is the (v_3,v_n)-lollipop.

The following theorem shows that the bound given by Theorem 5 is actually a sharp bound.

Theorem 7. *Let G be a* (v_s,v_n)*-lollipop and s is odd. Then* $\exp(G) = 2n - s - 1$.

Proof. Since $\ell = n - 5$, Theorem 5 implies that $\exp(G) \leq s + 2\ell - 1 = 2n - s - 1$. It remains to show that $\exp(G) \geq 2n - s - 1$. Notice that the smallest closed walk of odd length from v_n to itself is of length $2n - s$. This implies there is no closed walk of length $2n - s - 2$ from v_n to itself. Hence the $\exp(G) \geq 2n - s - 1$. ■

As a direct consequence of Theorem 7 we have the following corollary that gives a formula for exponents of cycles of length odd.

Corollary 8. *Let G be a cycle of length odd n. Then the* $\exp(G) = n - 1$.

We note that Theorem 7 and Corollary 8 give classes of primitive graphs that attain the upper bound given in Theorem 5. The following theorem gives a more general class of primitive graphs that attain the bound in Theorem 5. We employ the following terminologies. A *forest* is a disconnected graph which has no cycles. A *cycle-forest* is a connected graph with exactly one cycle. A typical example of a cycle forest is given by Figure 2.

Theorem 9. *Let G be a cycle-forest on n vertices with the cycle of length s. Then* $\exp(G) = s + 2\ell - 1$.

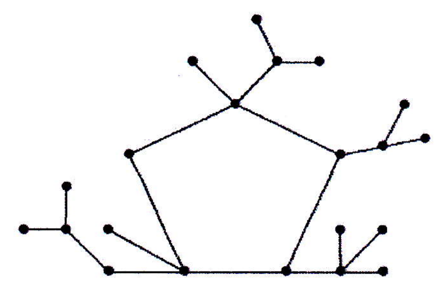

Fig. 2. A cycle forest.

Proof. Let C be the cycle in G of length s. Let a be a vertex in $G\backslash C$ and x be a vertex in C such that

$$\ell = \ell(p_{ax}) = \max_{u \in G\backslash C, c \in C} \{\ell(p_{uc})\}$$

Notice that for each pair of vertices u and v in G there is a uv-path p_{uv} with $\ell(p_{uv}) \leq s + 2\ell - 1$. Proposition 2 guarantees that for each pair of vertices u and v in G there is a uv-walk w_{uv} of length exactly $s + 2\ell - 1$. Hence, $\exp(G) \leq s + 2\ell - 1$. We note that the smallest closed walk w_{aa} of odd length is the walk that starts at a, moves to x along the path p_{ax}, moves around the cycle C once and back to x, and finally moves back to a along the path p_{ax}. The length of this walk is $s + 2\ell$. This implies there is no closed walk from a to a in G of length $s + 2\ell - 1$. Hence $\exp(G) \geq s + 2\ell - 1$. Now we can conclude that $\exp(G) = s + 2\ell - 1$. ∎

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