Proceedings of the 2nd IMT-GT Regional Conference on Mathematics, Statistics and Applications
Universiti Sains Malaysia

λ-BACKBONE COLORING NUMBERS OF SPLIT GRAPHS WITH TREE BACKBONES

A.N.M. Salman

Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung, Jalan Ganesa 10 Bandung 40132, Indonesia
msalman@math.itb.ac.id

Abstract. In the application area of frequency assignment graphs are used to model the topology and mutual interference between transmitters. The problem in practice is to assign a limited number of frequency channels in an economical way to the transmitter in such a way that interference is kept at an acceptable level. This has led to various different types of coloring problem in graphs. One of them is a λ-backbone coloring. Given an integer $\lambda \geq 2$, a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a λ-backbone coloring of (G, H) is a proper vertex coloring $V \rightarrow \{1, 2, \ldots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least λ. The λ-backbone coloring number $BBC_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \rightarrow \{1, 2, \ldots, \ell\}$. In this paper we consider the λ-backbone coloring of split graphs. A split graph is a graph whose vertex set can be partitioned into a clique (i.e. a set of mutually adjacent vertices) and an independent set (i.e. a set of mutually non adjacent vertices), with possibly edges in between. We determine sharp upper bounds for λ-backbone coloring numbers of split graphs with tree backbones.

1. Introduction

In [3] backbone colorings are introduced, motivated and put into a general framework of coloring problems related to frequency assignment. We refer to [3] and [2] for an overview of related research, but we repeat the relevant definitions here. For undefined terminology we refer to [1].

Let $G = (V, E)$ be a graph, where $V = V_G$ is a finite set of vertices and $E = E_G$ is a set of unordered pairs of two different vertices, called edges. A function $f : V \rightarrow \{1, 2, 3, \ldots\}$ is a *vertex coloring* of V if $|f(u) - f(v)| \geq 1$ holds for all edges $uv \in E$. A vertex coloring $f : V \rightarrow \{1, \ldots, k\}$ is called a *k-coloring*, and the *chromatic number* $\chi(G)$ is the smallest integer k for which there exists a k-coloring. A set $V' \subseteq V$ is *independent* if G does not contain edges with both end vertices in V'. By definition, a k-coloring partitions V into k independent sets $V_1, \ldots, V_k$.

Let H be a *spanning subgraph* of G, i.e., $H = (V_G, E_H)$ with $E_H \subseteq E_G$. Given an integer $\lambda \geq 2$, a vertex coloring f of G is a *λ-backbone coloring* of (G, H), if $|f(u) - f(v)| \geq \lambda$ holds for all edges $uv \in E_H$. The *λ-backbone coloring number* $\text{BBC}_\lambda(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a λ-backbone coloring $f : V \rightarrow \{1, \ldots, \ell\}$.

A *path* is a graph P whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_P = \{v_1v_2, \ldots, v_{n-1}v_n\}$. A *cycle* is a graph C whose vertices can be ordered into a sequence $v_1, v_2, \ldots, v_n$ such that $E_C = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$. A *tree* is a connected graph T that does not contain any cycles.

A *complete* graph is a graph with an edge between every pair of vertices. The complete graph on n vertices is denoted by K_n. A graph G is *complete p-partite* if its vertices can be partitioned into p nonempty independent sets $V_1, \ldots, V_p$ such that its edge set E is formed by all edges that have one end vertex in V_i and the other one in V_j for some $1 \leq i < j \leq p$.

A *star* S_q is a complete 2-partite graph with independent sets $V_1 = \{r\}$ and V_2 with $|V_2| = q$; the vertex r is called the *root* and the vertices in V_2 are called the *leaves* of the star S_q. In our context a *matching* M is a collection of pairwise disjoint stars that are all copies of S_1. We call a spanning subgraph H of a graph G

- a *tree backbone* of G if H is a (spanning) tree;
- a *star backbone* of G if H is a collection of pairwise disjoint stars;
- a *matching backbone* of G if H is a (perfect) matching.

Obviously, $\text{BBC}_\lambda(G, H) \geq \chi(G)$ holds for any backbone H of a graph G. In order to analyze the maximum difference between these two numbers the following values can be introduced.

$$\mathcal{T}_\lambda(k) = \max\{\text{BBC}_\lambda(G,T) \mid T \text{ is a tree backbone of } G, \text{ and } \chi(G)=k\}$$
$$\mathcal{S}_\lambda(k) = \max\{\text{BBC}_\lambda(G,S) \mid S \text{ is a star backbone of } G, \text{ and } \chi(G)=k\}$$
$$\mathcal{M}_\lambda(k) = \max\{\text{BBC}_\lambda(G,M) \mid M \text{ is a matching backbone of } G, \text{ and } \chi(G)=k\}.$$

For the case $\lambda = 2$, the behavior of the first values is determined in [3] as summarized in the following result.

Theorem 1.1. $\mathcal{T}_2(k) = 2k-1$ *for all* $k \geq 1$.

The above theorem shows the relation between the 2-backbone coloring number and the classical chromatic number in case the backbone is a tree. The 2-backbone coloring number roughly grow like $2k$, where $\chi = k$. In [4], we studied the other two cases: We first determined all values $\mathcal{S}_\lambda(k)$, and observed that they roughly grow like $(2-\frac{1}{\lambda})k$. Then we determined all values $\mathcal{M}_\lambda(k)$ and observed that they roughly grow like $(2-\frac{2}{\lambda+1})k$. Their precise behavior is summarized in the two following theorems.

Theorem 1.2. *For* $\lambda \geq 2$ *the function* $\mathcal{S}_\lambda(k)$ *takes the following values:*

(a) $\mathcal{S}_\lambda(2) = \lambda+1$;
(b) *for* $3 \leq k \leq 2\lambda-3$: $\mathcal{S}_\lambda(k) = \lceil\frac{3k}{2}\rceil + \lambda - 2$;
(c) *for* $2\lambda-2 \leq k \leq 2\lambda-1$ *with* $\lambda \geq 3$: $\mathcal{S}_\lambda(k) = k+2\lambda-2;\ \mathcal{S}_2(3) = 5$;
(d) *for* $k = 2\lambda$ *with* $\lambda \geq 3$: $\mathcal{S}_\lambda(k) = 2k-1$; $\mathcal{S}_2(4) = 6$;
(e) *for* $k \geq 2\lambda+1$: $\mathcal{S}_\lambda(k) = 2k - \lfloor\frac{k}{\lambda}\rfloor$.

Theorem 1.3. *For* $\lambda \geq 2$ *the function* $\mathcal{M}_\lambda(k)$ *takes the following values:*

(a) *for* $2 \leq k \leq \lambda$: $\mathcal{M}_\lambda(k) = \lambda+k-1$;
(b) *for* $\lambda+1 \leq k \leq 2\lambda$: $\mathcal{M}_\lambda(k) = 2k-2$;
(c) *for* $k = 2\lambda+1$: $\mathcal{M}_\lambda(k) = 2k-3$;
(d) *for* $k = t(\lambda+1)$ *with* $t \geq 2$: $\mathcal{M}_\lambda(k) = 2t\lambda$;
(e) *for* $k = t(\lambda+1)+c$ *with* $t \geq 2$, $1 \leq c < \frac{\lambda+3}{2}$: $\mathcal{M}_\lambda(k) = 2t\lambda+2c-1$;
(f) *for* $k = t(\lambda+1)+c$ *with* $t \geq 2$, $\frac{\lambda+3}{2} \leq c \leq \lambda$: $\mathcal{M}_\lambda(k) = 2t\lambda+2c-2$.

A *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e. a set of mutually adjacent vertices) and an *independent set* (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique and the size of a largest independent set in G are denoted by $\omega(G)$ and $\alpha(G)$, respectively. Split graphs were introduced by Hammer and Földes [7]; see also the book [6] by Golumbic. They form an interesting subclass of the class of perfect graphs. Hence, split graphs satisfy $\chi(G) = \omega(G)$.

The sharp upper bounds for the λ-backbone coloring numbers of split graphs with star or matching backbones are determined in [5] as follows.

Theorem 1.4. *Let* $\lambda \geq 2$ *and let* $G = (V,E)$ *be a split graph with* $\chi(G) = k \geq 2$. *For every star backbone* $S = (V, E_S)$ *of* G,

$$\text{BBC}_\lambda(G,S) \leq \begin{cases} k+\lambda & \text{if either } k=3 \text{ and } \lambda \geq 2 \text{ or } k \geq 4 \text{ and } \lambda = 2 \\ k+\lambda-1 & \text{in the other cases.} \end{cases}$$

The bounds are tight.

Theorem 1.5. *Let* $\lambda \geq 2$ *and let* $G = (V,E)$ *be a split graph with* $\chi(G) = k \geq 2$. *For every matching backbone* $M = (V, E_M)$ *of* G,

$$\text{BBC}_\lambda(G,M) \leq \begin{cases} \lambda+1 & if\ k=2 \\ k+1 & if\ k \geq 3 \text{ and } \lambda \leq \min\{\frac{k}{2}, \frac{k+5}{3}\} \\ k+2 & if\ k=9 \text{ or } k \geq 11 \text{ and } \frac{k+6}{3} \leq \lambda \leq \lceil\frac{k}{2}\rceil \\ \lceil\frac{k}{2}\rceil+\lambda & if\ k=3,5,7 \text{ and } \lambda \geq \lceil\frac{k}{2}\rceil \\ \lceil\frac{k}{2}\rceil+\lambda+1 & if\ k=4,6 \text{ or } k \geq 8 \text{ and } \lambda \geq \lceil\frac{k}{2}\rceil+1. \end{cases}$$

The bounds are tight.

In this paper we study the special case of λ-backbone colorings of split graphs with tree backbones. In the next section we present sharp upper bounds for the λ-backbone coloring numbers of split graphs with tree backbones.

2. Split graphs with tree backbones

In 2003 Broersma et al. [3] determined sharp upper bounds for the λ-backbone coloring numbers of split graphs along trees for $\lambda = 2$ as summarized in the following theorem.

Theorem 2.1. *Let $G = (V, E)$ be a split graph with $\chi(G) = k \geq 1$. For every tree backbone $T = (V, E_T)$ of G,*

$$\text{BBC}_2(G,T) \leq \begin{cases} 1 & \text{if } k = 1 \\ 3 & \text{if } k = 2 \\ k+2 & \text{if } k \geq 3. \end{cases}$$

The bound is tight.

We study λ-backbone colorings of split graphs along trees for other values of λ and generalize the result in Theorem 2.1 as follows.

Theorem 2.2. *Let $\lambda \geq 2$ and let $G = (V, E)$ be a split graph $\chi(G) = k \geq 1$. For every tree backbone T of G,*

$$\text{BBC}_\lambda(G,T) \leq \begin{cases} 1 & \text{if } k = 1 \\ 1+\lambda & \text{if } k = 2 \\ k+\lambda & \text{if } k \geq 3. \end{cases}$$

The bounds are tight.

Proof of the upper bounds. Let $G = (V, E)$ be a split graph with a spanning tree $T = (V, E_T)$. Let C and I be a partition of V such that C with $|C| = k$ is a clique of maximum size, and such that I is an independent set. Since split graphs are perfect, $\chi(G) = \omega(G) = k$. The case $k = 1$ is trivial. If $k = 2$ then G is bipartite, and we use colors 1 and $\lambda + 1$. For $k \geq 3$, we consider the restriction of the tree T to the vertices in C, and we distinguish two cases.

In the first case, the restriction of T to C forms a star $K_{1,k-1}$. Let $v_1, \ldots, v_{k-1}$ denote the $k-1$ leaves of this star, and let v_k denote its center. For $i = 1, \ldots, k-1$ we color v_i with color i, and we color v_k with color $k + \lambda - 1$. This yields a λ-backbone coloring for the vertices in C. All vertices $u \in I$ are leaves in the tree T. Any vertex $u \in I$ with $uv_k \notin E_T$ can be safely colored with color $k + \lambda$. It remains to consider vertices $u \in I$ with $uv_k \in E_T$. In the graph G, such a vertex u is nonadjacent to at least one of the vertices $v_1, \ldots, v_{k-1}$, say to vertex v_j (otherwise, the clique C could be augmented by vertex u and would not be of maximum size as we assumed). In this case we may color u with color j.

In the second case, the restriction of T to C does not form a star. In this case the restriction of T to C has a proper 2-coloring $C = C_1 \cup C_2$ with $|C_1| = a \geq |C_2| = b \geq 2$. Then there exist a vertex $x \in C_1$ and a vertex $y \in C_2$ for which $xy \notin E_T$. Let $v_1, \ldots, v_a = x$ be an enumeration of the vertices in C_1, and let $y = v_{a+1}, \ldots, v_{a+b}$ be an enumeration of the vertices in C_2. For $i = 1, \ldots, a$ we color vertex v_i with color $i + 1$. For $i = 1, \ldots, b$ we color vertex v_{a+i} with color $a + \lambda + i - 1$. This yields a λ-backbone coloring of C with colors in $\{2, \ldots, k + \lambda - 1\}$. We color each vertex $u \in I$ with color

$$\begin{cases} k+\lambda & \text{if } uv \in E_T \text{ and } v \in C_1 \\ 1 & \text{if } uv \in E_T \text{ and } v \in C_2. \end{cases}$$

This yields a λ-backbone $(k + \lambda)$-coloring of (G, T), since the colors of a vertex v_i with $i \in \{1, \ldots, a\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $k + \lambda - (i + 1) \geq k + \lambda - (k - 2 + 1) > \lambda$, and since the colors of a vertex v_i with $i \in \{a + 1, \ldots, b\}$ and of any vertex $u \in I$ such that $uv_i \in E_T$ have distance at least $a + \lambda + i - 1 - 1 \geq k/2 + \lambda - 1 \geq \lambda$.

Proof of the tightness of the bounds. The cases $k = 1$ and $k = 2$ are trivial. For $k \geq 3$, we consider a split graph with a clique of k vertices $v_1, \ldots, v_k$ and with an independent set of $(k - 2)(k - 1)/2$ vertices $u_{i,j}$ with $1 \leq i < j \leq k - 1$. Every vertex $u_{i,j}$ is adjacent to all vertices v_s with $s \neq i$. The tree backbone T contains the $k - 1$ edges $v_k v_s$ with $1 \leq s \leq k - 1$. The vertices $u_{i,j}$ form the leaves of T; in the tree, vertex $u_{i,j}$ is adjacent only to v_j. Clearly, $\chi(G) = k$.

Suppose to the contrary that $\text{BBC}_\lambda(G, T) \leq k + \lambda - 1$, and consider such a backbone coloring. The vertices $v_1, \ldots, v_k$ in the clique must be colored with k pairwise distinct colors. Since they form a star, either vertex v_k has color 1, and colors $2, \ldots, \lambda$ are not used on the clique, or vertex v_k has color $k + \lambda - 1$, and colors $k, \ldots, k + \lambda - 2$ are not used on the clique. Both cases are symmetric, and we assume without loss of generality that v_k has color

$k+\lambda-1$ and that colors $k,\ldots,k+\lambda-2$ are not used on the clique. Let v_i be the vertex that has color $k-2$, and let v_j be the vertex that has color $k-1$. The vertex $u_{i,j}$ is adjacent to all clique vertices except v_i; hence, it could only be colored with color $k-2$ or with a color in $\{k,\ldots,k+\lambda-2\}$. But these λ colors are forbidden for $u_{i,j}$, since in the tree backbone it is adjacent to vertex v_j with color $k-1$. Since there is no feasible color for $u_{i,j}$, we arrive at the desired contradiction.

3. Acknowledgments

This research was supported by the Research Fund of Institut Teknologi Bandung, Program: Riset Unggulan ITB 2006.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications,* Macmillan, London and Elsevier, New York (1976).
[2] H.J. Broersma, A general framework for coloring problems: old results, new results and open problems, *Lecture Notes in Computer Science* **3330**(2004), 65–79.
[3] H.J. Broersma, F.V. Fomin, P.A. Golovach, and G.J. Woeginger, Backbone colorings for networks, *Lecture Notes in Computer Science* **2880**(2003), 131–142.
[4] H.J. Broersma, J. Fujisawa, L. Marchal, D. Paulusma, A.N.M. Salman, and K. Yoshimoto, λ-Backbone colorings along pairwise disjoint stars and matchings, *submitted* (2006).
[5] H.J. Broersma, L. Marchal, D. Paulusma, and A.N.M. Salman, Improved upper bounds for λ-backbone colorings along matchings and stars, *submitted* (2006).
[6] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs,* Academic Press, New York (1980).
[7] P.L. Hammer and S. Földes, Split graphs, *Congressus Numerantium* **19**(1977), 311–315.