

FULL IDENTIFICATION OF IDEMPOTENTS IN BINARY ABELIAN GROUP RINGS

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Abstract. Every code in the latest study of group ring codes is a submodule that has a generator. Study reveals that each of these binary group ring codes can have multiple generators that have diverse algebraic properties. However, idempotent generators get the most attention as codes with an idempotent generator are easier to determine its minimal distance. We have fully identify all idempotents in every binary cyclic group ring algebraically using basis idempotents. However, the concept of basis idempotent constrained the flexibilities of extending our work into the study of identification of idempotents in non-cyclic groups. In this paper, we extend the concept of basis idempotent into idempotent that has a generator, called a generated idempotent. We show that every idempotent in an abelian group ring is either a generated idempotent or a finite sum of generated idempotents. Lastly, we show a way to identify all idempotents in every binary abelian group ring algebraically by fully obtain the support of each generated idempotent.

Key words and Phrases: idempotent, generated idempotent, group ring code, binary abelian group ring.

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Abstrak. Setiap kode dalam kajian kode gelanggang grup didefinisikan sebagai suatu submodul yang mempunyai pembangun. Telah diketahui bahwa setiap kode gelanggang grup biner dapat memiliki banyak pembangun yang kaya akan sifat-sifat aljabar. Dari sekian banyak pembangun ini, pembangun idempoten adalah jenis pembangun yang paling menarik, karena kode dengan pembangun idempoten mudah ditentukan jarak minimumnya. Telah diidentifikasi semua idempoten dari setiap gelanggang grup biner yang siklik secara aljabar dengan menggunakan basis idempoten. Akan tetapi, konsep basis idempoten membatasi fleksibilitas untuk memperluas kajian ke arah identifikasi idempoten untuk kasus tak-siklik. Di dalam paper ini, diperluas konsep basis idempoten menjadi idempoten yang memiliki pembangun, yang disebut dengan idempoten yang dibangun (*generated idempotent*). Kemudian, ditunjukkan bahwa idempoten pada suatu gelanggang grup yang komutatif adalah berbentuk *generated idempotent* atau jumlah hingga dari *generated idempotent*. Lebih jauh, ditunjukkan juga suatu cara untuk mengidentifikasi semua idempoten di setiap gelanggang grup yang komutatif secara aljabar dengan cara menentukan semua *support* dari setiap *generated idempotent*.

Kata kunci: Idempoten, *generated idempotent*, kode gelanggang grup, gelanggang grup biner komutatif.

1. INTRODUCTION

In this paper, G is referred as a finite abelian group. Let F_q be a finite field of q elements. A group ring of G over F_q is the set $F_qG = \{ \sum_{g \in G} a_g g \mid a_g \in F_q \}$. The support of a group ring element $u = \sum_{g \in G} a_g g \in F_qG$ is denoted as $\text{supp}(u) = \{g \in G \mid a_g \neq 0\}$. Furthermore, if $u \in F_qG$ satisfying $u^2 = u$, then u is called an idempotent in F_qG . Every group ring F_qG has at least two idempotents, namely the trivial idempotents, which are 0 and 1.

Since 1967, S.D. Berman introduced the classical notion of F_qG codes, which are basically ideals of F_qG [1]. For decades, having idempotents as generators of F_qG code has been widely proven to be an essence in finding the codes' minimum distance [3, 9, 10, 11]. In 2006, Ted Hurley and Paul Hurley introduced a modern approach to study F_qG codes, which are submodules of the F_qG [4]. Toward this direction, the resultant F_qG codes are categorized into two main families, namely the zero-divisor codes and unit-derived codes, with respect to whether their generator is a zero divisor or unit. Some researches show that the zero-divisor codes are good sources of self-dual codes [2, 4, 6, 8]. In fact, the non-trivial idempotents are also zero-divisors. This inspires us to study the potential of having idempotent generators for zero-divisor codes in determining their properties. However, in an arbitrary group ring, it is not an easy task to identify those idempotents from a large pool of group ring elements.

In [7], we have fully identify all idempotents in every F_2C_n algebraically, where C_n is a cyclic group, by using our concept of basis idempotents. We show

that every F_2C_n will have a unique idempotent, e_L with largest support size. We introduce a way to partition the support of e_L into the supports of some idempotents which we defined as basis idempotents. The set of linear combination of the basis idempotents over F_2 is then shown to be exactly the set of all idempotents in F_2C_n . However; the definition of our basis idempotents constrained the flexibilities of extending our work into the study of identification of all idempotents in F_2G , where G is non-cyclic.

In this paper, we extend our concept of basis idempotents in F_2C_n into our concept of generated idempotents in F_2G . In Section 2, we introduce the generated idempotents as well as their important properties. We show that every idempotent in F_2G is either a generated idempotent or a finite sum of some generated idempotents and there exists a unique largest finite sum of generated idempotents in F_2G . In Section 3, we show the largest odd order subgroup of G forms the support of the largest finite sum of generated idempotents in F_2G . We also prove that the support of all non-zero generated idempotents in F_2G can be obtained by partitioning the support of the largest finite sum of generated idempotents and thus we identify all the idempotents in F_2G .

2. GENERATED IDEMPOTENTS

Recall that G is referred as a finite abelian group. Let $I(G)$ denote the set of all idempotents in F_2G . The closure property on the addition of the set is affirmed by the following proposition.

Proposition 2.1. *The set $I(G)$ is closed under addition.*

PROOF. Let $e_1, e_2 \in I(G)$, then $(e_1 + e_2)^2 = e_1^2 + 2e_1e_2 + e_2^2 = e_1 + e_2$.

Next we introduce a special type of permutation that help in differentiating elements in $I(G)$ from F_2G .

Definition 2.2. *Let $a \in F_2G$. Define $\varphi_a : \text{supp}(a) \rightarrow G$ such that $\varphi_a(g) = g^2$. If φ_a is a bijection on $\text{supp}(a)$, we called φ_a a support permutation of a .*

Let $a = \sum_{i=1}^n g_{j_i} \in F_2G$. Note that $a^2 = a$ if and only if $(\sum_{i=1}^n g_{j_i})^2 = \sum_{i=1}^n g_{j_i}$. Since $\text{char}(F_2) = 2$, $(\sum_{i=1}^n g_{j_i})^2 = \sum_{i=1}^n g_{j_i}^2 + \sum_{i < k} g_{j_i}g_{j_k} + \sum_{i > k} g_{j_i}g_{j_k} = \sum_{i=1}^n g_{j_i}^2 + \sum_{i < k} g_{j_i}g_{j_k} + \sum_{i < k} g_{j_i}g_{j_k} = \sum_{i=1}^n g_{j_i}^2 + 2 \sum_{i < k} g_{j_i}g_{j_k} = \sum_{i=1}^n g_{j_i}^2$. Thus, we have $(\sum_{i=1}^n g_{j_i})^2 = \sum_{i=1}^n g_{j_i}$ if and only if $\sum_{i=1}^n g_{j_i}^2 = \sum_{i=1}^n g_{j_i}$. Thus, $a^2 = a$ if and only if $\sum_{i=1}^n g_{j_i}^2 = \sum_{i=1}^n g_{j_i}$. Hence we have proved the following result:

Proposition 2.3. *Let $a \in F_2G$. Then $a \in I(G)$ if and only if φ_a is a support permutation of a .*

In addition, the support permutation φ_a of each idempotent $e \in I(G)$ is unique.

Corollary 2.4. *Let $g \in G$. If $\text{ord}(g) = 2$, then g does not contain in the support of any idempotent.*

PROOF. Let $e \in I(G)$. Suppose that $g \in \text{supp}(e)$ and $\text{ord}(g) = 2$.

Case 1: $1 \in \text{supp}(e)$. Then, $\varphi_e(g) = \varphi_e(1) = 1$, contrary to φ_e is the support permutation of e .

Case 2: $1 \notin \text{supp}(e)$. But $1 \in \text{Im}(\varphi_e)$ as $\varphi_e(g) = 1$, contrary to φ_e is the support permutation of e .

Thus, $g \notin \text{supp}(e)$.

Suppose that $e \in I(G)$ has the support permutation φ_e which is a cycle of length n in $S_{|G|}$ where $n = |\text{supp}(e)|$. Then it is clear that for any arbitrary $g \in \text{supp}(e)$,

$$e = g + \varphi_e(g) + \varphi_e(\varphi_e(g)) + \cdots + \varphi_e^{n-1}(g) = \sum_{i=1}^n g^{2^i-1}$$

This observation motivates the following definition:

Definition 2.5. *Let $e \in I(G)$ such that its support permutation φ_e is a cycle of length n in $S_{|G|}$ where $n = |\text{supp}(e)|$. Then any element $g \in \text{supp}(e)$ is called a generator of e and e is called a generated idempotent, denote as $e = \langle g \rangle$.*

Note that 0 and 1 are the two trivial generated idempotents in every F_2G . On the other hand, it is obvious by Definition 2.5 that a non-trivial idempotent in F_2G is not a generated idempotent if its support contains the identity $1 \in G$. In addition, each generated idempotent $e \in F_2G$ has exactly n many distinct generators if φ_e is a cycle of length n . Below are two immediate results indicated by Definition 2.5:

Proposition 2.6. *Let $x \in G$. Then $x \in \text{supp}(e)$ where $e = \langle g \rangle$ if and only if $x = g^{2^j}$ for some unique $j \in \{0, 1, 2, \dots, |\text{supp}(e)| - 1\}$.*

Proposition 2.7. *Let $x \in G$. Then $x \in \text{supp}(e)$ where $e = \langle g \rangle$ if and only if $g = x^{2^j}$ for some unique $j \in \{0, 1, 2, \dots, |\text{supp}(e)| - 1\}$.*

Also, it is clear that both Definition 2.5 and Proposition 2.6 implies the following result otherwise it will contradict Proposition 2.3

Proposition 2.8. *Let $x \in G$. If $x \in \text{supp}(e)$ where $e = \langle g \rangle$, then $\text{ord}(x) = \text{ord}(g)$.*

Since a cycle can never be written as a product of shorter disjoint cycles, it is obvious that the support of any generated idempotent will not contain any other idempotent's. More detailed, we show that the supports of two distinct generated idempotents are mutually disjoint.

Proposition 2.9. *Let $e_1 = \langle g_1 \rangle$ and $e_2 = \langle g_2 \rangle$ be two distinct generated idempotents in F_2G , then $\text{supp}(e_1) \cap \text{supp}(e_2) = \emptyset$.*

PROOF. Suppose that $\text{supp}(e_1) \cap \text{supp}(e_2) \neq \emptyset$. Then by Proposition 2.6, for every $x \in \text{supp}(e_1) \cap \text{supp}(e_2)$, $x = g_1^{2^{j_1}}$ and $x = g_2^{2^{j_2}}$ for some unique $j_1 \in \{0, 1, 2, \dots, |\text{supp}(e_1)| - 1\}$ and $j_2 \in \{0, 1, 2, \dots, |\text{supp}(e_2)| - 1\}$. On the other hand, both Proposition 2.8 and Definition 2.5 indicate that $\text{ord}(x) = \text{ord}(g_1) = \text{ord}(g_2)$ and $\langle x \rangle = \langle g_1 \rangle = \langle g_2 \rangle$, which then lead to $e_1 = e_2$, a contradiction. Thus, $\text{supp}(e_1) \cap \text{supp}(e_2) = \emptyset$.

The notion of generated idempotent will completely formalize all non-zero idempotents in as illustrated in Theorem 2.10 below.

Theorem 2.10. *Every non-zero idempotent in F_2G is either a generated idempotent or a finite sum of generated idempotents.*

PROOF. Let e be a non-zero idempotent in F_2G with its support permutation φ_e . If $e = 1$, then e is a trivial generated idempotent. Let e be a non-trivial idempotent. Then $|\text{supp}(e)| > 1$ by Corollary 2.4.

Case 1: If φ_e is a cycle with length equal to $|\text{supp}(e)|$, then e is a generated idempotent.

Case 2: Suppose that φ_e is a product of disjoint cycles, says $\varphi_e = \varphi_1 \varphi_2 \cdots \varphi_k$ such that the sum of all the length of φ_i for $i = 1, 2, \dots, k$ is equal to $|\text{supp}(e)|$. Let $e_i \in F_2G$ such that $\text{supp}(e_i) \subset \text{supp}(e)$ and having φ_i as its support permutation.

By Definition 2.5, each e_i is a generated idempotent. By Proposition 2.9, $e = \sum_{i=1}^k e_i$ and thus it is a finite sum of the generated idempotents e_i for $i = 1, 2, \dots, k$.

Proposition 2.1 and Theorem 2.10 indicate that knowing all the generated idempotents in F_2G guarantees the full identification of the elements in $I(G)$. However, identifying all the generated idempotents in F_2G is not an easy task for G with large cardinality. Since the support of all generated idempotents are mutually disjoint and $I(G)$ is closed under addition, there exists a largest finite sum of all generated idempotents in F_2G .

Definition 2.11. *The finite sum of all generated idempotents in F_2G is called the largest idempotent in F_2G , denoted as e_{LG} .*

Clearly, e_{LG} is unique for each F_2G . Hence, we have the following result:

Theorem 2.12. *The largest idempotent exists and is unique in every F_2G .*

In the next section, the support of e_{LG} in every F_2G will be identified and partitioned into disjoint supports of all non-zero generated idempotents.

3. IDENTIFICATION OF ALL IDEMPOTENTS

This section is devoted to our main results. We first prove that $e_{L_G} = 1$ for every F_2G with $|G| = 2^\alpha$, $\alpha \in \mathbb{Z}^+ \cup \{0\}$. Next, we identify every e_{L_G} where $|G|$ has an odd factor greater than one. For these cases, we further introduce a method to partition the support of e_{L_G} in order to fully obtain all the non-zero generated idempotents in F_2G and thus all the idempotents in F_2G .

It is obvious that the generator of every generated idempotent in F_2G will also be a generator of a cyclic subgroup of G . Next, we further proof that the cyclic subgroup generated by a generator of a non-trivial generated idempotent in F_2G must be non-trivial and of odd order.

Theorem 3.1. *For $g \in G$, $\langle g \rangle$ is a non-trivial generated idempotent if and only if $\text{ord}(g)$ is odd and $\text{ord}(g) > 1$.*

PROOF. Suppose that $\text{ord}(g)$ is odd and $\text{ord}(g) > 1$, then there exist a smallest $k \in \mathbb{Z}^+$ and $k > 1$ such that $g^{2^k} = g$. Thus, $\{g, g^2, g^4, \dots, g^{2^{k-1}}\}$ forms a support of a non-trivial generated idempotent $\langle g \rangle$ by Proposition 2.6.

Conversely, suppose there exists a non-trivial generated idempotent $\langle g \rangle$ such that $\text{ord}(g) = 2k$ for some $k \in \mathbb{Z}^+$. By Corollary 2.4, we have $k > 1$. Also by Definition 2.5, then there exists a smallest $j \in \mathbb{Z}^+$ and $j > 1$ such that $g = g^{2^j}$. Thus, we have $1 \equiv 2^j \pmod{2k}$. Solving the congruence equation, we yield $2^j - 1 = 2lk$, for some $l \in \mathbb{Z}^+$. This is a contradiction since the left hand side of the equation is always odd as $j > 1$ and the right hand side of the equation is always even. Hence, $\text{ord}(g)$ must be odd and greater than one.

Theorem 3.1 above further suggests that the support of every non-trivial generated idempotent is contained in a non-trivial odd order subgroup of G . Note that if $|G| = 2^\alpha$ for some $\alpha \in \mathbb{Z}^+$, then G will not have any non-trivial odd order subgroup. Thus, we have the following result:

Corollary 3.2. *if $|G| = 2^\alpha$ for some $\alpha \in \mathbb{Z}^+$, then F_2G has only trivial idempotents.*

Next, we continue to study the idempotents in where has an odd factor greater than one. Suppose that $|G|$ has exactly r many distinct odd prime factors for some $r \in \mathbb{Z}^+$. Then, $|G| = 2^\alpha p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$ for some $\alpha \in \mathbb{Z}^+ \cup \{0\}$ and some $\beta_1, \beta_2, \dots, \beta_r \in \mathbb{Z}^+$. By the Fundamental Theorem of Finite Abelian Group, $G \simeq H \otimes K$, where $|H| = 2^\alpha$ and $|K| = p_1^{\beta_1} p_2^{\beta_2} \dots p_r^{\beta_r}$. It is clear that this K is the unique non-trivial subgroup of G having the largest odd order. From now on, we denote this subgroup K of G having the largest odd order as H_L and also if H is a subgroup of G , we denote the group ring element $\sum_{g \in H} g$ as \bar{H} .

Theorem 3.3. *If $|G|$ has an odd factor greater than one, then $e_{L_G} = \bar{H}_L$.*

PROOF. Since $|H_L|$ is odd, we have $\bar{H}_L^2 = \bar{H}_L \bar{H}_L = (\sum_{g \in H_L} g) \bar{H}_L = \sum_{g \in H_L} (g \bar{H}_L) = |H_L| \bar{H}_L = \bar{H}_L$. Thus \bar{H}_L is an idempotent. Next we show that \bar{H}_L is a finite sum

of generated idempotents. Note that $1 \in \text{supp}(\bar{H}_L) = H_L$ and since $1^{2^i} = 1$ for any $i \in \mathbb{Z}^+$, we know that $\varphi_{\bar{H}_L}$ will not be a cycle and thus by Theorem 2.10, \bar{H}_L is a finite sum of at least two generated idempotents. Lastly, we claim that the support of every generated idempotent contains in H_L . Suppose that there exists $g \in G - \{1\}$ such that $\langle g \rangle$ is a non-trivial generated idempotent with $g \notin H_L$. Let H be the cyclic subgroup of G generated by g . Then by Theorem 3.1, H must be odd. Note that $H \cap H_L$ is a proper subgroup of H as $g \notin H_L$. Hence, HH_L is another subgroup of G having odd order $|HH_L|$ that is greater than $|H_L|$ as $|HH_L| = \frac{|H||H_L|}{|H \cap H_L|}$ and $\frac{|H|}{|H \cap H_L|} > 1$. This is a contrary to H_L is the largest odd order subgroup of G . Thus, every generated idempotent's support is contained in H_L . By Proposition 2.9 and Theorem 2.10, we have shown that $e_{L_G} = \bar{H}_L$.

By the Fundamental Theorem of Finite Abelian Group, $H_L \simeq \bigotimes_{i=1}^k C_i$ for some $k \in \mathbb{Z}^+$, where $C_i = \langle a_i | a_i^{n_i} = 1 \rangle$ and n_i is odd for every $i \in \{1, 2, \dots, k\}$. Then, every group element $g \in H_L$ can be written in the form of $g = \prod_{i=1}^k a_i^{m_i}$ where $0 \leq m_i < n_i$ for every $i \in \{1, 2, \dots, k\}$. Recall the group isomorphism $\phi : H_L \rightarrow \bigotimes_{i=1}^k \mathbb{Z}_{n_i}$ such that $\phi(\prod_{i=1}^k a_i^{m_i}) = (m_1, m_2, \dots, m_k)$. In order to partition H_L to obtain the support of all the disjoint generated idempotents, it is the same as partitioning $\text{Im}(\phi)$ into the isomorphic form of the generated idempotents' support.

Let $(m_1, m_2, \dots, m_k), (r_1, r_2, \dots, r_k) \in \text{Im}(\phi)$, define a relation such that $(m_1, m_2, \dots, m_k) \sim (r_1, r_2, \dots, r_k)$ if and only if there exists $j \in \mathbb{Z}^+ \cup \{0\}$ such that for every $i \in \{1, 2, \dots, k\}$, $m_i = 2^j r_i \pmod{n_i}$. It can be checked easily that \sim is an equivalence relation on $\text{Im}(\phi)$. Thus, the set $\text{Im}(\phi)$ is partitioned by \sim and each of the partition is named by using definition below.

Definition 3.4. *The set $K_{(m_1, m_2, \dots, m_k)} = \{(r_1, r_2, \dots, r_k) \in \text{Im}(\phi) | (r_1, r_2, \dots, r_k) \sim (m_1, m_2, \dots, m_k)\}$ is called the 2-generalized cyclotomic coset of $\text{Im}(\phi)$ containing (m_1, m_2, \dots, m_k) .*

Next, we show that each 2-generalized cyclotomic coset of $\text{Im}(\phi)$ induces a generated idempotent in F_2G .

Theorem 3.5. *Suppose that K_1, K_2, \dots, K_l are all the 2-generalized cyclotomic cosets of $\text{Im}(\phi)$, then there are exactly l many generated idempotents in F_2G , namely e_1, e_2, \dots, e_l such that each $e_i = \sum_{(m_1, m_2, \dots, m_k) \in K_i} \prod_{j=1}^k a_j^{m_j}$.*

PROOF. Let $K_{(m_1, m_2, \dots, m_k)}$ be a 2-generalized cyclotomic coset. If $(r_1, r_2, \dots, r_k) \in K_{(m_1, m_2, \dots, m_k)}$, then there exists $j \in \mathbb{Z}^+ \cup \{0\}$ such that for every $i \in \{1, 2, \dots, k\}$, $m_i = 2^j r_i \pmod{n_i}$. That is, $\prod_{i=1}^k a_i^{m_i} = \prod_{i=1}^k a_i^{2^j r_i} = (\prod_{i=1}^k a_i^{r_i})^{2^j}$. Then by Proposition 2.7, we have $\prod_{i=1}^k a_i^{r_i} \in \text{supp}(\langle \prod_{i=1}^k a_i^{m_i} \rangle)$ if and only if $\phi(\prod_{i=1}^k a_i^{r_i}) \in K_{(m_1, m_2, \dots, m_k)}$.

Hence, $\phi^{-1}(K_{(m_1, m_2, \dots, m_k)}) = \text{supp}(\langle \prod_{i=1}^k a_i^{m_i} \rangle)$. Thus, each of the group ring element $e_i = \sum_{(m_1, m_2, \dots, m_k) \in K_i} \prod_{j=1}^k a_j^{m_j}$ is a generated idempotent for every $i \in \{1, 2, \dots, k\}$.

On the other hand, let $e = \langle \prod_{i=1}^k a_i^{m_i} \rangle$. Then by Definition 2.5 and Proposition 2.7, for every $x = \prod_{i=1}^k a_i^{r_i} \in \text{supp}(e)$, there exists a unique $j \in \{0, 1, 2, \dots, k-1\}$ such that $\prod_{i=1}^k a_i^{m_i} = x^{2^j} = (\prod_{i=1}^k a_i^{r_i})^{2^j} = \prod_{i=1}^k a_i^{2^j r_i}$. This is equivalent to for every $x = \prod_{i=1}^k a_i^{r_i} \in \text{supp}(e)$, there exists a unique $j \in \{0, 1, 2, \dots, k-1\}$ such that for every $i \in \{1, 2, \dots, k\}$, $m_i = 2^j r_i \pmod{n_i}$. Thus, $\text{supp}(e) = \{ \prod_{i=1}^k a_i^{r_i} \mid (r_1, r_2, \dots, r_k) \in K_{(m_1, m_2, \dots, m_k)} \}$.

Note that result below further assure that the generated idempotents are generalization from our basis idempotents in [7].

Corollary 3.6. *Let $\{e_1, e_2, \dots, e_l\}$ be the set of all generated idempotents in F_2G . Then, $\{b_1 e_1 + b_2 e_2 + \dots + b_l e_l \mid b_i \in F_2\}$ is the set of all idempotents in F_2G . Moreover, we have $|I(G)| = 2^l$.*

The following examples illustrate the full identification of idempotents in some F_2G :

Example 3.7. *Consider the binary group ring F_2G where $G = C_1 \otimes C_2$ such that $C_1 = \langle a_1 \mid a_1^3 = 1 \rangle$ and $C_2 = \langle a_2 \mid a_2^9 = 1 \rangle$. Clearly, G itself is exactly H_L . By Theorem 3.3, G forms the support of the largest idempotent e_{LG} . Using Definition 3.4, we yield all the 2-generalized cyclotomic cosets of $\phi(G)$.*

$$\begin{aligned} K_{(0,0)} &= \{(0, 0)\}, K_{(1,0)} = \{(1, 0), (2, 0)\}, \\ K_{(0,1)} &= \{(0, 1), (0, 2), (0, 4), (0, 5), (0, 7), (0, 8)\}, K_{(1,3)} = \{(1, 3), (2, 6)\}, \\ K_{(0,3)} &= \{(0, 3), (0, 6)\}, K_{(1,1)} = \{(1, 1), (2, 2), (1, 4), (2, 5), (1, 7), (2, 8)\}, \\ K_{(2,3)} &= \{(2, 3), (1, 6)\}, K_{(1,2)} = \{(2, 1), (1, 2), (2, 4), (1, 5), (2, 7), (1, 8)\}. \end{aligned}$$

By Theorem 3.5, we yield the corresponding generated idempotents:

$$\begin{aligned} e_1 &= 1, e_2 = a_1 + a_1^2, e_3 = a_2 + a_2^2 + a_2^4 + a_2^5 + a_2^7 + a_2^8, e_4 = a_1 a_2^3 + a_1^2 a_2^6, \\ e_5 &= a_2^3 + a_2^6, e_6 = a_1 a_2 + a_1^2 a_2^2 + a_1 a_2^4 + a_1^2 a_2^5 + a_1 a_2^7 + a_1^2 a_2^8, e_7 = a_1^2 a_2^3 + a_1 a_2^6, \\ e_8 &= a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_2^4 + a_1 a_2^5 + a_1^2 a_2^7 + a_1 a_2^8. \end{aligned}$$

The set of all idempotents in F_2G is $I(G) = \{ \sum_{i=1}^8 b_i e_i \mid b_i \in F_2 \}$.

Example 3.8. Consider the binary group ring F_2G where $G = C_1 \otimes C_2 \otimes C_3$ such that $C_1 = \langle a_1 | a_1^2 = 1 \rangle$, $C_2 = \langle a_2 | a_2^3 = 1 \rangle$ and $C_3 = \langle a_3 | a_3^6 = 1 \rangle$. Clearly, H_L is $C_2 \otimes C'_3$ which is of order 9, where C'_3 is the subgroup of C_3 generated by a_3^2 . Thus, the largest idempotent e_{L_G} has support $C_2 \otimes C'_3$. Using Definition 3.4 to partition $\phi(C_2 \otimes C'_3)$, we have the following 2-generalized cyclotomic cosets:

$K_{(0,0)} = \{(0,0)\}$, $K_{(1,0)} = \{(1,0), (2,0)\}$, $K_{(0,1)} = \{(0,1), (0,2)\}$,
 $K_{(1,1)} = \{(1,1), (2,2)\}$, $K_{(1,2)} = \{(1,2), (2,1)\}$. Each of them induces a generated idempotent by Theorem 3.5, which are $e_1 = 1$, $e_2 = a_2 + a_2^2$, $e_3 = a_3^2 + a_3^4$, $e_4 = a_2a_3^2 + a_2^2a_3^4$ and $e_5 = a_2a_3^4 + a_2^2a_3^2$ respectively. The set of all idempotents in F_2G , is exactly $I(G) = \left\{ \sum_{i=1}^5 b_i e_i \mid b_i \in F_2 \right\}$.

4. CONCLUDING REMARKS

The intention to discover the potential of zero-divisor codes having idempotent generator triggered us to identify all the idempotents in binary abelian group rings. In this paper, we introduced the notion of generated idempotent and proved some important properties of these idempotents that enable the identification of all the idempotents in any arbitrary binary abelian group ring. However, the fact that every idempotent is a generated idempotent or finite sum of generated idempotents is generally not true for binary non-abelian group rings.

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