BOUNDS ON ENERGY AND LAPLACIAN ENERGY OF GRAPHS

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Abstract. Let $G$ be simple graph with $n$ vertices and $m$ edges. The energy $E(G)$ of $G$, denoted by $E(G)$, is defined to be the sum of the absolute values of the eigenvalues of $G$. In this paper, we present two new upper bounds for energy of a graph, one in terms of $m, n$ and another in terms of largest absolute eigenvalue and the smallest absolute eigenvalue. The paper also contains upper bounds for Laplacian energy of graph.

Key words and Phrases: Adjacency matrix, Laplacian matrix, Energy of graph, Laplacian energy of graph.

Abstrak. Misalkan $G$ adalah graf sederhana dengan $n$ titik dan $m$ sisi. Energi $E(G)$ dari $G$, dinotasikan dengan $E(G)$, didefinisikan sebagai jumlahan dari nilai mutlak dari nilai-nilai eigen $G$. Pada paper ini, kami menyatakan dua batas atas baru untuk energi dari graf, satu batas dalam suku $m$, $n$ dan batas yang lain dalam suku nilai eigen mutlak terbesar dan terkecil. Paper ini juga memuat batas atas untuk energi Laplace dari graf.

Kata kunci: Matriks ketetanggaan, matriks Laplace, energi dari graf, energi Laplace dari graf.

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1. Introduction

The concept of energy of a graph was introduced by I. Gutman [6] in the year 1978. Let $G$ be a graph with $n$ vertices $\{v_1, v_2, ..., v_n\}$ and $m$ edges and $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $A$, assumed in non-increasing order, are the eigenvalues of the graph $G$. The energy $E(G)$ of $G$ is defined to be the sum of the absolute values of the eigenvalues of $G$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on the mathematical aspects of the theory of graph energy see the papers [2, 3, 8] and the references cited there in. The basic properties including various upper and lower bounds for energy of a graph have been established in [10] and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 9]. The bounds for eigenvalues of graph can be found in [1, 13].

Definition 1.1. Let $G$ be a graph with $n$ vertices and $m$ edges. The Laplacian matrix of the graph $G$, denoted by $L = (L_{ij})$, is a square matrix of order $n$ whose elements are defined as

\[
L_{ij} = \begin{cases} 
-1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\
0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \\
d_i & \text{if } i = j
\end{cases}
\]

where $d_i$ is the degree of the vertex $v_i$.

Eigenvalues of $L$ is called eigenvalues of $G$.

Definition 1.2. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Laplacian eigenvalues of $G$. Laplacian energy $LE(G)$ of $G$ is defined as $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$.

The matrix $L$ is positive semi-definite and therefore its eigenvalues are non-negative. The least eigenvalue is always equal to zero. The second largest eigenvalue is called the algebraic connectivity of $G$. The basic properties including various upper and lower bounds for Laplacian energy have been established in [7, 11, 12, 13].

2. Main Results

2.1. Energy of graph. We denote the decreasing order of the the absolute value of eigenvalues of $G$ by $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The following are the elementary results that follows from this notation.

(1) $\rho_i = |\lambda_k|$ for some $k$

(2) $\rho_i \geq \lambda_i$ for all $i$

(3) $E(G) = \sum_{i=1}^{n} \rho_i$
(4) \( \rho_n \leq \sum_{i=1}^{n} \rho_i = E(G) \)

(5) By Cauchy-Schwarz inequality

\[
\left( \sum_{i=1}^{n} \lambda_i \rho_i \right)^2 \leq \left( \sum_{i=1}^{n} \rho_i^2 \right) \left( \sum_{i=1}^{n} \lambda_i^2 \right)
\]

\[\sum_{i=1}^{n} \lambda_i \rho_i \leq \sqrt{(2m)(2m)} \]

Therefore \( \sum_{i=1}^{n} \lambda_i \rho_i \leq 2m \), equality holds if \( \rho_i = \lambda_i \).

(6) Let \( G \) and \( H \) be any two graphs with same \( n \) vertices each. Let their number of edges be respectively \( m_1 \) and \( m_2 \). If \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \) and \( \rho'_1 \geq \rho'_2 \geq \ldots \geq \rho'_n \) are their the absolute value of eigenvalues then

\[
\sum_{i=1}^{n} \rho_i \rho'_i \leq \sqrt{\left( \sum_{i=1}^{n} \rho_i^2 \right) \left( \sum_{i=1}^{n} \rho'_i^2 \right)} \leq \sqrt{(2m_1)(2m_2)}
\]

\[\therefore \sum_{i=1}^{n} \rho_i \rho'_i \leq 2\sqrt{m_1 m_2} \]

(7) Since \( \lambda_1 \) is always positive, so \( \rho_1 = \lambda_1 \geq \frac{2m}{n} \)

(8) Since \( n\rho_n^2 \leq \rho_1^2 + \rho_2^2 + \ldots + \rho_n^2 = 2m \) which implies \( \rho_n \leq \sqrt{\frac{2m}{n}} \)

**Theorem 2.1.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \) be the the absolute value of eigenvalues of \( G \) then \( \rho_n \leq \sqrt{\frac{2m(n-1)}{n}} \).

**Proof.** We know that \( E(G) = \sum_{i=1}^{n} \rho_i \) and \( \sum_{i=1}^{n} \rho_i^2 = 2m \)

Since \( \rho_n \leq \rho_i \) \( \forall i \therefore \rho_n \leq \sum_{i=1}^{n-1} \rho_i \)

By Cauchy Schwarz inequality

\[
\left( \sum_{i=1}^{n-1} \rho_i \right)^2 \leq \sum_{i=1}^{n-1} \rho_i^2 \sum_{i=1}^{n-1} \rho_i^2
\]

\[= (n-1) \sum_{i=1}^{n-1} \rho_i^2 \]

\[\Rightarrow \sum_{i=1}^{n-1} \rho_i^2 \geq \frac{1}{(n-1)} \left( \sum_{i=1}^{n} \rho_i \right)^2 \]
\[ 2m - \rho_n^2 \geq \frac{1}{(n-1)} \left( \sum_{i=1}^{n-1} \rho_i \right)^2 \]
\[ \geq \frac{1}{(n-1)} \rho_n^2 \]
\[ \Rightarrow \rho_n \leq \sqrt{\frac{2m(n-1)}{n}} \]

which is an upper bound for the smallest absolute eigenvalue of the graph \( G \). \( \Box \)

**Theorem 2.2.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \) be the the absolute value of eigenvalues of \( G \). If \( \rho_1 \) is repeated \( k \) times then

\[ \rho_1 \leq \frac{1}{k(p-1)} \left( \sqrt{2mkp(p-1)} - \sum_{i=k+1}^{kp} \rho_i \right) \text{ where } kp \leq n \text{ and } p \neq 1, k \neq 0. \]

**Proof.** Let \( H = \bigcup_{k} K_p \cup \left(K_{n-kp}\right)^c \) where \( kp \leq n \)

That is \( H \) is the union of graphs \( K_p \), repeated \( k \) times and a graph \((K_{n-kp})^c\).

The number of vertices of \( H \) is \( n \) and the number of edges is \( \frac{kp(p-1)}{2} \).

Its the absolute value of eigenvalues spectrum is

\[ \left( \begin{array}{ccc}
    p-1 & 1 & 0 \\
    k & k(p-1) & (n-kp) \\
\end{array} \right) . \]

By Cauchy Schwarz inequality

\[ \rho_1(p-1) + \ldots + \rho_{k}(p-1) + \rho_{k+1}(1) + \ldots + \rho_{kp}(1) + \rho_{kp+1}(0) + \ldots + \rho_{n}(0) \leq 2\sqrt{m\frac{kp(p-1)}{2}} \]

But \( \rho_1 = \rho_2 = \ldots = \rho_k \)

\[ (p-1)k\rho_1 + \sum_{i=k+1}^{kp} \rho_i \leq 2\sqrt{m\frac{kp(p-1)}{2}} \]

\[ \rho_1 \leq \frac{1}{k(p-1)} \left( \sqrt{2mkp(p-1)} - \sum_{i=k+1}^{kp} \rho_i \right) \text{ Here } (p \neq 1, k \neq 0) \]

\[ \Box \]

**Corollary 2.3.** If \( kp = n \), then by the above theorem

\[ (n-k)\rho_1 + \sum_{i=k+1}^{n} \rho_i \leq \sqrt{\frac{2mn(n-k)}{k}} \]

\[ (n-k)\rho_1 + E(G) - kp_1 \leq \sqrt{\frac{2mn(n-k)}{k}} \]
\[(n - 2k)\rho_1 + E(G) \leq \sqrt{\frac{2mn(n - k)}{k}}\]

\[E(G) \leq \sqrt{\frac{2mn(n - k)}{k}} - (n - 2k)\rho_1\]

Also if \(p = 2\) and \(2k = n\) then the upper bound for energy of graph is

\[E(G) \leq \sqrt{\frac{2mn(2k - k)}{k}}\]

\[E(G) \leq \sqrt{2mn}.

**Corollary 2.4.** If \(kp = n - 1\), then we get the following result.

\[E(G) - \rho_n \leq \sqrt{\frac{2m(n - 1)(n - 1 - k)}{k}} - (n - 1 - 2k)\rho_1\]

\[E(G) \leq \sqrt{\frac{2m(n - 1)(n - 1 - k)}{k}} - (n - 1 - 2k)\rho_1 + \rho_n.

Also if \(p = 2\) and \(2k = n - 1\) then the upper bound for energy of graph is

\[E(G) \leq \sqrt{2m(n - 1)} + \rho_n.

**Corollary 2.5.** If \(k = 1\), then \(E(G) \leq \sqrt{2mn(n - 1)} - (n - 2)\rho_1\) for \(p = n\). and \(E(G) \leq \sqrt{2m(n - 1)(n - 2)} - (n - 3)\rho_1 + \rho_n\) for \(p = n - 1\).

**Corollary 2.6.** Since \(\rho_1 \geq \frac{2m}{n}\) and \(\rho_n \leq \sqrt{\frac{2m}{n}}\) we get new upper bound for energy of graph in term of \(m\) and \(n\)

\[E(G) \leq \sqrt{\frac{2mn(n - k)}{k}} - (n - 2k)\frac{2m}{n}\text{ for }pk = n.

\[E(G) \leq \sqrt{\frac{2m(n - 1)(n - 1 - k)}{k}} - (n - 1 - 2k)\frac{2m}{n} + \sqrt{\frac{2m}{n}}\text{ for }pk = n - 1.

**Corollary 2.7.** For a \(r\)-regular graph \(m = \frac{rn}{2}\) and \(\rho_1 = r\) we have the following upper bound

\[E(G) \leq n\sqrt{\frac{r(n - k)}{k}} - (n - 2k)r\text{ for }pk = n.

\[E(G) \leq \sqrt{\frac{rn(n - 1)(n - 1 - k)}{k}} - (n - 1 - 2k)r + \sqrt{r}\text{ for }pk = n - 1.
Theorem 2.8. Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$ be the the absolute value of eigenvalues of $G$. If $\rho_1$ is repeated $k$ times then

$$\rho_1 \leq \frac{1}{k} \left( 2\sqrt{mk} - \sum_{i=k+1}^{2k} \rho_i \right). \quad (k \neq 0)$$

Proof. Here we compare the absolute value of eigenvalues of $G$ with absolute eigenvalue of the graph $H = \bigcup_k K_{p,q}$.

Select $p$ and $q$ such that $n = k(p + q)$. The number of vertices of $H$ is $n$ and the number of edges is $kpq$. Its the absolute value of eigenvalues spectrum are

$$\begin{pmatrix} \sqrt{pq} & 0 \\ 2k & (n - 2k) \end{pmatrix}.$$

By Cauchy Schwarz inequality

$$\rho_1 \sqrt{pq} + \ldots + \rho_k \sqrt{pq} + \rho_{k+1} \sqrt{pq} + \ldots + \rho_{2k} \sqrt{pq} + \rho_{2k+1}(0) + \ldots + \rho_n(0) \leq 2\sqrt{mkpq}$$

But $\rho_1 = \rho_2 = \ldots = \rho_k$

$$\therefore \rho_1 k \sqrt{pq} + \sqrt{pq} \sum_{i=k+1}^{2k} \rho_i \leq 2\sqrt{mkpq}$$

$$\rho_1 + \sum_{i=k+1}^{2k} \rho_i \leq 2\sqrt{mk}$$

$$\rho_1 \leq \frac{1}{k} \left( 2\sqrt{mk} - \sum_{i=k+1}^{2k} \rho_i \right).$$

\[\square\]

Corollary 2.9. If $p = q = 1$ and $2k = n$ then

$$\rho_1 k + \sum_{i=k+1}^{n} \rho_i \leq 2 \sqrt{\frac{m \cdot n}{2}}$$

i.e., $E(G) \leq \sqrt{2mn}$.

Corollary 2.10. If $p = q = 1$ and $2k = n - 1$ then

$$\rho_1 k + \sum_{i=k+1}^{n-1} \rho_i \leq 2 \sqrt{\frac{m \cdot (n - 1)}{2}}$$

$$\Rightarrow E(G) - \rho_n \leq \sqrt{2m(n - 1)}$$

i.e., $E(G) \leq \sqrt{2m(n - 1)} + \rho_n$
i.e., $E(G) \leq \sqrt{2m(n-1)} + \sqrt{\frac{2m}{n}}$.

**Corollary 2.11.** For $k = 1$, $\rho_1 + \rho_2 \leq 2\sqrt{m}$.

Using the above corollary we obtain another bound for energy of graphs.

**Theorem 2.12.** Let $G$ be a graph with $n$ vertices and $m$ edges and $2m \geq n$. If the first absolute eigenvalue, $\rho_1$ not repeated then $E(G) \leq \sqrt{m(2 + \sqrt{2n-4})}$

**Proof.** Cauchy Schwarz inequality for $(n-2)$ terms is

$$\left(\sum_{i=3}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=3}^{n} a_i^2\right) \left(\sum_{i=3}^{n} b_i^2\right)$$

Put $a_i = \rho_i$ and $b_i = 1$

$$\sum_{i=3}^{n} \rho_i \leq \sqrt{\left(\sum_{i=3}^{n} \rho_i^2\right) \left(\sum_{i=3}^{n} 1\right)}$$

Thus

$$E(G) - (\rho_1 + \rho_2) \leq \sqrt{(2m - (\rho_1^2 + \rho_2^2))(n-2)}$$

But $\rho_1 + \rho_2 \leq 2\sqrt{m}$ \therefore $E(G) \leq 2\sqrt{m} + \sqrt{n - 2\sqrt{(2m - (\rho_1^2 + \rho_2^2))}}$

We maximize the function $f(x, y) = 2\sqrt{m} + \sqrt{n - \frac{2}{2m}}\sqrt{(2m - (x^2 + y^2))}$

Then $f_x = \frac{-\sqrt{n - \frac{2}{2m}}x}{\sqrt{(2m - (x^2 + y^2))}}$ and $f_y = \frac{-\sqrt{n - \frac{2}{2m}}y}{\sqrt{(2m - (x^2 + y^2))}}$

For maxima value $f_x = 0$ and $f_y = 0$ which implies $(x, y) \equiv (0, 0)$

$$f_{xx} = \frac{-\sqrt{n - \frac{2}{2m}}(2m - y^2)}{(2m - (x^2 + y^2))^2}, f_{yy} = \frac{-\sqrt{n - \frac{2}{2m}}(2m - x^2)}{(2m - (x^2 + y^2))^2}, f_{xy} = \frac{\sqrt{n - 2xy}}{(2m - (x^2 + y^2))^2}$$

At $(x, y) \equiv (0, 0)$, $f_{xx} = -\frac{\sqrt{n - \frac{2}{2m}}}{2m}, f_{yy} = -\frac{\sqrt{n - \frac{2}{2m}}}{2m}, f_{xy} = 0$ and

$$\Delta = f_{xx}f_{yy} - (f_{xy})^2 = \frac{\sqrt{n - \frac{2}{2m}}}{2m}$$

Thus $f(x, y)$ attains maximum value at $(0, 0)$ \therefore $f(0, 0) = \sqrt{m(2 + \sqrt{2n-4})}$

$$E(G) \leq \sqrt{m(2 + \sqrt{2n-4})}.$$ 

**2.2. Laplacian energy of graph.** Analogous to the bounds for energy of graphs, now we obtain bounds for Laplacian energy of graphs.

**Theorem 2.13.** Let $G$ and $H$ are two graphs with $n$ vertices each. Let their number of edges be respectively be $m_1$ and $m_2$. If $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ represent
absolute Laplacian eigenvalues of $G$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ eigenvalues of $H$ then
\[
\sum_{i=1}^{n} \sigma_i \lambda_i \leq \sqrt{(2m_2) \left( 2m_1 + \sum_{i=1}^{n} \left( d_i(G) \right)^2 \right)}
\]
where $d_i(G)$ is the degree of the vertex $v_i$.

Proof. By Cauchy Schwarz inequality
\[
\sum_{i=1}^{n} \sigma_i \lambda_i \leq \sqrt{\left( \sum_{i=1}^{n} \sigma_i^2 \right) \left( \sum_{i=1}^{n} \lambda_i^2 \right)}
\]
But
\[
\sum_{i=1}^{n} \sigma_i^2 = (2m_1 + \sum_{i=1}^{n} \left( d_i(G) \right)^2)
\]
\[
\therefore \sum_{i=1}^{n} \sigma_i \lambda_i \leq \sqrt{(2m_2) \left( 2m_1 + \sum_{i=1}^{n} \left( d_i(G) \right)^2 \right)}.
\]

\[\Box\]

**Theorem 2.14.** Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ be the absolute Laplacian eigenvalues of $G$. If $\sigma_1$ is repeated $k$ times then
\[
\sigma_1 \leq \frac{1}{k(p-1)} \left( \sqrt{\left( 2m + \sum_{i=1}^{n} \left( d_i(G) \right)^2 \right) \frac{kp(p-1)}{2} - \sum_{i=k+1}^{kp} \sigma_i} \right)
\]
where $kp \leq n$, $k \neq 0, p \neq 1$

Proof. Let $H = \left( \bigcup_k K_p \right) \cup \left( K_{n-kp} \right)^c$ where $kp \leq n$

That is $H$ is union of graphs $K_p$, repeated $k$ times and a graph $(K_{n-kp})^c$.

The number of vertices of $H$ is $n$ and the number of edges is $\frac{kp(p-1)}{2}$.
Its the absolute value of eigenvalues spectrum is
\[
\begin{pmatrix}
  p-1 & 1 & 0 \\
  k & k(p-1) & (n-kp) \\
\end{pmatrix}.
\]

By Cauchy Schwarz inequality
\[
\sigma_1(p-1) + \sigma_2(p-1) + \ldots + \sigma_k(p-1) + \sigma_{k+1}(1) + \sigma_{k+2}(1) + \ldots + \sigma_{kp}(1) + \sigma_{kp+1}(0) + \ldots + \sigma_n(0) \leq \sqrt{\left( 2m + \sum_{i=1}^{n} \left( d_i(G) \right)^2 \right) \frac{kp(p-1)}{2}}
\]

But $\sigma_1 = \sigma_2 = \ldots = \sigma_k$
\[
(p-1)k \sigma_1 + \sum_{i=k+1}^{kp} \sigma_i \leq \sqrt{\left( 2m + \sum_{i=1}^{n} \left( d_i(G) \right)^2 \right) \frac{kp(p-1)}{2}}
\]
Bounds on Energy and Laplacian Energy of Graphs

\[ k\sigma_1 \leq \frac{1}{p-1} \left( \sqrt{2m + \sum_{i=1}^{n} (d_i(G))^2} \right)^{k(p-1)} \frac{1}{2} - \sum_{i=k+1}^{p} \sigma_i \]

\[ \sigma_1 \leq \frac{1}{k(p-1)} \left( \sqrt{2m + \sum_{i=1}^{n} (d_i(G))^2} \right)^{k(p-1)} \frac{1}{2} - \sum_{i=k+1}^{p} \sigma_i \].

**Corollary 2.15.** If \( kp = n \) then by the above theorem

\[ (n-k)\sigma_1 + \sum_{i=k+1}^{kp} \sigma_i \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{n(n-k)}{k}} \]

\[ (n-k)\sigma_1 + LE(G) - k\sigma_1 \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{n(n-k)}{k}} \]

\[ (n-2k)\sigma_1 + LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{n(n-k)}{k}} \]

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{(2k-k)(n-k)}{k}} \]

Also if \( p = 2 \) and \( 2k = n \) then the upper bound for Laplacian energy of graph

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{n(n-k)}{k}} \]

**Corollary 2.16.** If \( kp = n - 1 \) we get the following result.

\[ LE(G) - \sigma_n \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{(n-1)(n-1-k)}{k}} - (n-1 - 2k)\sigma_1 \]

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{(n-1)(n-1-k)}{k}} - (n-1 - 2k)\sigma_1 + \sigma_n \]

Also if \( p = 2 \) and \( 2k = n - 1 \) then we get the following upper bound for Laplacian energy of graph

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) \frac{(n-1)(2k-k)}{k} + \sigma_n} \]

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) (n-1) + \sigma_n} \]

**Corollary 2.17.** If \( k = 1 \) then the upper bounds changes to

\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right) n(n-1) - (n-2)\sigma_1} \text{ for } p = n \]
\[ LE(G) \leq \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right)(n-1)(n-2) - (n-3)\sigma_1 + \sigma_n} \text{ for } p = n - 1. \]

**Theorem 2.18.** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( \sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \) be the absolute Laplacian eigenvalues of \( G \). If \( \sigma_1 \) is repeated \( k \) times then
\[
\sigma_1 \leq \frac{1}{k} \left( \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right)2k} - \sum_{i=k+1}^{2k} \sigma_i \right) \quad (k \neq 0).
\]

**Proof.** Here we compare absolute Laplacian eigenvalues of \( G \) with absolute eigenvalue of graph \( H = \left( \bigcup_k K_{p,q} \right) \).

Select \( p \) and \( q \) such that \( n = k(p + q) \). The number of vertices of \( H \) is \( n \) and the number of edges is \( kpq \). Its the absolute value of eigenvalues spectrum is
\[
\left( \begin{array}{cc}
\sqrt{pq} & 0 \\
2k & (n-2k)
\end{array} \right).
\]

By Cauchy Schwarz inequality
\[
\sigma_1 \sqrt{pq} + \ldots + \sigma_k \sqrt{pq} + \sigma_{k+1} \sqrt{pq} + \ldots + \sigma_{2k} \sqrt{pq} + \sigma_{2k+1}(0) + \ldots + \sigma_n(0) \leq \sqrt{(2m + \sum_{i=1}^{n} (d_i(G))^2)^2} 2kpq
\]

But \( \sigma_1 = \sigma_2 = \ldots = \sigma_k \)

\[
\therefore \sigma_1 k \sqrt{pq} + \sqrt{pq} \sum_{i=k+1}^{2k} \sigma_i \leq \left( 2m + \sum_{i=1}^{n} (d_i(G))^2 \right) 2kpq
\]

\[
\sigma_1 k + \sum_{i=k+1}^{2k} \sigma_i \leq \left( 2m + \sum_{i=1}^{n} (d_i(G))^2 \right) 2k
\]

\[
\sigma_1 \leq \frac{1}{k} \left( \sqrt{\left(2m + \sum_{i=1}^{n} (d_i(G))^2\right)2k} - \sum_{i=k+1}^{2k} \sigma_i \right). \quad \square
\]

**Corollary 2.19.** If \( 2k = n \) then by above theorem
\[
LE(G) \leq \sqrt{(2m + \sum_{i=1}^{n} (d_i(G))^2)n}.
\]

**Corollary 2.20.** If \( 2k = (n - 1) \) then by above theorem
\[
LE(G) \leq \sqrt{(2m + \sum_{i=1}^{n} (d_i(G))^2)(n - 1) + \sigma_n}.
\]

**References**


Bounds on Energy and Laplacian Energy of Graphs