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SOME PROPERTIES OF MULTIPLICATION MODULES

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Abstract. Let M be an R-module. The module M is called multiplication if for any submodule N of M we have $N = IM$, where I is an ideal of R. In this paper we state some basic properties of submodules of these modules. Also, we study the relationship between the submodules of a multiplication R-module M and ideals of ring R. Finally, by definition of semiprime submodule, we state some properties of radical submodules of multiplication modules.

Key words and Phrases: Multiplication module, prime submodule, primary submodule.

Abstrak. Diberikan ring R dan diketahui M adalah R-modul. Modul M disebut modul multiplikasi jika untuk setiap submodul N di M memenuhi sifat $N = IM$, untuk suatu ideal I di R. Dalam paper ini diberikan beberapa sifat dasar submodul-submodul dalam modul multiplikasi. Selain itu diberikan juga hubungan antara submodul di dalam modul multiplikasi M atas R dan ideal-ideal di dalam ring R. Dengan menggunakan definisi submodul semiprima, dihasilkan juga beberapa sifat submodule radikal dalam modul multiplikasi.

Kata kunci: Modul multiplikasi, submodul prima, submodul utama.

1. Introduction

In this paper all rings are commutative with identity and all modules over rings are unitary. Let K and N be submodules of an R-module M, we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R. Let N be a proper submodule of an R-module M, then N is called a prime submodule of M,

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if for every $r \in R$, $x \in M$; $rx \in N$ we have $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. The set of all prime submodules of M is denoted by $Spec(M)$ and for a submodule N of M, $rad(N) = \bigcap_{L \in Spec(M), N \subseteq L} L$. If no prime submodule of M contains N, we write $rad(N) = M$. Also the set of all maximal submodules of M is denoted by $Max(M)$ and $RadM = \bigcap_{P \in Max(M)} P$. Also we recall that if I is an ideal of a ring R, then radical of I, i.e., $r(I)$ is defined as $\{r \in R \mid \exists k \in \mathbb{N} ; r^k \in I\}$. Now, let $\underline{a}$ be an ideal of a ring R and $\underline{a} = \bigcap_{i=1}^{l} q_i$, where $r(q_i) = p_i$ is a normal primary decomposition of $\underline{a}$, then $ass(\underline{a}) = \{p_1, ..., p_l\}$.
An R-module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It can be shown that $N = (N : M)M$.

2. Definitions and Results

Lemma 2.1. *Let R be a non-trivial ring and let M be a multiplication R-module. Then $IM \neq M$ for any proper ideal I of R.*

PROOF. Let I be an arbitrary proper ideal of R, then there exists a maximal ideal $\underline{m}$ of R such that $I \subseteq \underline{m}$. We remind that $R_{\underline{m}} = S^{-1}R$ and $M_{\underline{m}} = S^{-1}M$ are quotient ring of R and quotient module respectively where $S = R - \underline{m}$.
We show that $\underline{m}M \neq M$. By [1, Lemma 2(i)], $M_{\underline{m}}$ is a multiplication $R_{\underline{m}}$-module and also by [3, Theorem 2.5], $(\underline{m}R_{\underline{m}})M_{\underline{m}}$ is the only maximal submodule of $M_{\underline{m}}$. Thus $(\underline{m}R_{\underline{m}})M_{\underline{m}} = (\underline{m}M)_{\underline{m}} \neq M_{\underline{m}}$ and hence $((\underline{m}M)_{\underline{m}} : M_{\underline{m}}) = \underline{m}R_{\underline{m}}$. Therefore $\underline{m}M \neq M$.
Now since $IM \subseteq \underline{m}M \neq M$, therefore $IM \neq M$ for every proper ideal I of R.

Lemma 2.2. *Let M be a multiplication R-module, then $pM \in Spec(M)$ and $(pM : M) = p$ for every $(0) \neq p \in Spec(R)$.*

PROOF. Let M be a multiplication R-module and let $(0) \neq p \in Spec(R)$ be arbitrary. By Lemma 2.1, $pM \neq M$. We show that $(pM : M) = p$.
Let $r \in (pM : M)$ be arbitrary then $rM \subseteq pM$ and hence we have:

$$(rM)_p \subseteq (pM)_p \Longrightarrow \frac{r}{1}M_p \subseteq (pM)_p \Longrightarrow \frac{r}{1} \in ((pM)_p : M_p).$$

But by [1, Lemma 2(i)], M_p is a multiplication R_p-module and by [3, Theorem 2.5], $(pM)_p$ is the only maximal submodule of M_p and so $((pM)_p : M_p) = pR_p$. Thus $r \in p$ and hence $(pM : M) = p$.
Now by [2, Corollary 2], pM is a primary submodule of M and also by [5, Proposition 1], $pM \in Spec(M)$. The proof is now completed.

Corollary 2.3. *Let R be an arbitrary ring and let M be a multiplication R-module. Then there exists a bijection between non-zero prime ideals of R and non-zero prime submodules of M.*

PROOF. We show that for every $N \in Spec(M)$, $N = pM$ where $p \in Spec(R)$. First, we show that, if M is a multiplication R-module and N is a submodule of M, then $N = (N : M)M$. Since M is a multiplication R-module, hence there exists an ideal I of R such that $N = IM$. From this, we have $I \subseteq (N : M)$, and $N = IM \subseteq (N : M)M \subseteq N$, hence $N = (N : M)M$. Second, let $N \in Spec(M)$. Since M is a multiplication R-module, $N = (N : M)M$ and by [5, Proposition 1], $(N : M) \in Spec(R)$.
Now we define $\psi : Spec(M) \longrightarrow Spec(R)$ by $\psi(pM) = (pM : M)$ for any non-zero prime ideal p of R. Clearly ψ is well defined and by Lemma 2.2, ψ is a bijection.

Proposition 2.4. *Let M be a faithful multiplication R-module. Then every proper direct summand of M is prime. Hence M is indecomposable.*

PROOF. By [3, Lemma 4.1], M is torsion-free and by [5, Result 1], every direct summand of M is a prime submodule.
Now we show that M is indecomposable. If $M = M_1 \oplus M_2$ where $M_1, M_2 \neq 0$ then by the current form of the proposition, M_1 is a p-prime for some ideal p of R. Thus $M_1 = pM = pM_1 \oplus pM_2$. Hence $pM_2 = 0$. Since M is torsion-free and $M_2 \neq 0$, we have $p = (0)$ and hence $M_1 = 0$, a contradiction.

Proposition 2.5. *Let M be a multiplication R-module. Then for every submodule IM of M, if $IM \subseteq pM$ where $p \in Spec(R)$, then $I \subseteq p$.*

PROOF. Let $IM \subseteq pM$ for $I \trianglelefteq R$ and $p \in Spec(R)$. Since $I \subseteq (IM : M) \subseteq (pM : M)$, then by Lemma 2.2, $I \subseteq p$.

Corollary 2.6. *If M is a faithful multiplication R-module then M is finitely generated.*

PROOF. Since M is a multiplication R-module, hence by Lemma 2.1, $M \neq IM$ for every proper ideal of I of R. Now, since M is a faithful multiplication R-module, then by [3, Theorem 3.1 part (i), (iv)], M is finitely generated.

Corollary 2.7. *Let M be a faithful multiplication R-module. Then for every ideal I of R, $(IM : M) = I$.*

PROOF. Let M is a faithful multiplication R-module, then by Corollary 2.6, M is a finitely generated R-module. Now, let $(IM : M) = q$ where I and q be ideals of R. Since $(IM : M) = q$, $qM \subseteq IM$ and by [3, Theorem 3.1 part (ii)], $q \subseteq I$. Now, since $I \subseteq (IM : M) = q$, therefore $I = (IM : M)$.

Lemma 2.8. *If M is a faithful multiplication R-module. Then there exists a bijection between ideals of R and submodules of M.*

PROOF. Since M is a multiplication R-module, hence for every submodule N of M there exists an ideal I of R such that $N = IM$ and by Corollary 2.7, $(N : M) =$

$(IM : M) = I$. Now we define $\psi : M \longrightarrow R$ by $\psi(IM) = (IM : M)$ for any ideal I of R. Obviously ψ is well defined and also ψ is an epimorphism. Now let N_1, N_2 be submodules of M, then there exist $I_1, I_2 \trianglelefteq R$ such that $N_1 = I_1M$ and $N_2 = I_2M$. If $\psi(N_1) = \psi(N_2)$ then $(I_1M : M) = (I_2M : M)$ and by Corollary 2.7, $I_1 = I_2$. Therefore ψ is a bijection.

Corollary 2.9. *Let M be a Noetherian multiplication R-module. Then R satisfies the ascending chain condition on prime ideals.*

Proof. Let $p_1 \subseteq p_2 \subseteq p_3 \subseteq ...$ be an ascending chain of prime ideals of R. Then $p_1M \subseteq p_2M \subseteq p_3M \subseteq ....$ But, M is a Noetherian R-module, hence there exists submodule (by [3, Theorem 2.5 part (i)], specially a maximal submodule) N of M such that $p_1M \subseteq p_2M \subseteq p_3M \subseteq ... \subseteq N$. But M is a multiplication R-module, hence by [3, Theorem 2.5 part (ii)], there exists a maximal ideal $\underline{m}$ of R such that $N = \underline{m}M$. So we have $p_1M \subseteq p_2M \subseteq p_3M \subseteq ... \subseteq \underline{m}M$ and hence $(p_1M : M) \subseteq (p_2M : M) \subseteq (p_3M : M) \subseteq ... \subseteq (\underline{m}M : M)$. Now by Lemma 2.2, $p_1 \subseteq p_2 \subseteq p_3 \subseteq ... \subseteq \underline{m}$. The proof is now completed.

Corollary 2.10. *Let R be an arbitrary ring and let M be a multiplication R-module. Then $Ann_R(M) \subseteq p$ for each $(0) \neq p \in Spec(R)$.*

Proof. By the Lemma 2.2, $pM \in Spec(M)$ for every $(0) \neq p \in Spec(R)$. Therefore by [3, Corollary 2.11 part (i), (iii)], $Ann_R(M) \subseteq p$.

We recall that in the following lemma $J(R)$ and $\underline{n}_R$ denote the Jacobson radical and nilradical of R, respectively.

Lemma 2.11. *Let R be a ring and M a multiplication R-module. Then $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda} p_\lambda)M$ for any non-empty collection of non-zero prime ideals p_λ $(\lambda \in \Lambda)$ of R. Also if R is a ring which is not an integral domain then $\bigcap_{0\neq P\in Spec(M)} P = \underline{n}_R M$ and $RadM = J(R)M$.*

Proof. Let M be a multiplication R-module and let p_λ $(\lambda \in \Lambda)$ be any non-empty collection of non-zero prime ideals of R. By [3, Corollary 1.7], $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda}[p_\lambda + Ann_R(M)])M$. But by Corollary 2.10, $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda} p_\lambda)M$. By Lemma 2.2, $\bigcap_{0\neq P\in Spec(M)} P = \bigcap_{(0)\neq p\in Spec(R)}(pM)$ and also by above we have $\bigcap_{(0)\neq p\in Spec(R)}(pM) = \underline{n}_R M$. So $\bigcap_{0\neq P\in Spec(M)} P = \underline{n}_R M$. Also by Lemma 2.2, $RadM = \bigcap_{\underline{m}\in Max(R)}(\underline{m}M)$ and by above $\bigcap_{\underline{m}\in Max(R)} \underline{m}M = J(R)M$. Hence $RadM = J(R)M$.

Lemma 2.12. *Let R be a ring and M a multiplication R-module. Let IM be an arbitrary non-zero proper submodule of M for some ideal I of R. Then $rad(IM) = (radI)M$ and $(rad(IM) : M) = radI$, where $radI = r(I)$.*

Proof. It is easy to show that $rad(IM) = \bigcap_{p\in v(I)}(pM)$ (we recall that $v(I) = \{p \in Spec(R) |\ I \subseteq p\}$). By Lemma 2.11, $rad(IM) = (radI)M$ and consequently $(rad(IM) : M) = radI$.

We recall the following definition from [6].

Definition 2.13. *A proper submodule N of an R-module M is said to be semiprime in M, if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring R is an R-module by itself, a proper ideal I of R is semiprime if for every ideals J and K of R, $J^2K \subseteq I$ implies that $JK \subseteq I$.*

Definition 2.14. *There exists another definition of semiprime submodules in [4] as follows:*
A proper submodule N of the R-module M is semiprime if whenever $r^k m \in N$ for some $r \in R,\ m \in M$ and positive integer k, then $rm \in N$.
By [7, Remark 2.6], we see that this definition is equivalent to Definition 2.13.

Definition 2.15. *Let M be an R-module and $N \leq M$. The envelope of the submodule N is denoted by $E_M(N)$ or simply by $E(N)$ and is defined as $E(N) = \{x \in M \mid \exists r \in R,\ a \in M;\ x = ra \text{ and } r^n a \in N \text{ for some positive integer } n\}$.*

The envelope of a submodule is not a submodule in general.
Let M be an R-module and $N \leq M$. If there exists a semiprime submodule of M which contains N, then the intersection of all semiprime submodules containing N is called *semi-radical* of N and is denoted by $S - rad_M N$, or simply $S - radN$. If there is no semiprime submodule containing N, then we define $S - radN = M$, in particular $S - radM = M$.
We say that M *satisfies the radical formula*, or M (s.t.r.f) if for every $N \leq M$, $radN = \langle E(N) \rangle$. Also we say that M *satisfies the semi-radical formula*, or M (s.t.s.r.f) if for every $N \leq M$, $S - radN = \langle E(N) \rangle$. Now let $x \in E(N)$ and P be a semiprime submodule of M containing N. Then $x = ra$ for some $r \in R,\ a \in M$ and for some positive integer n, $r^n a \in N$. But $r^n a \in P$ and since P is semiprime we have $ra \in P$. Hence $E(N) \subseteq P$. We see that $E(N) \subseteq \bigcap P$ (P is a semiprime submodule containing N). So $\langle E(N) \rangle \subseteq S - radN$. On the other hand, since every prime submodule of M is clearly semiprime, we have $S - radN \subseteq radN$. We conclude that $\langle E(N) \rangle \subseteq S - radN \subseteq radN$ and as a result if M (s.t.r.f) then it is also (s.t.s.r.f).

Lemma 2.16. *Let R be a ring and let M be a multiplication R-module. Then every proper submodule of M is a radical submodule, i.e., $radN = N$.*

Proof. By [3, Theorem 2.12], $radN = rad(N : M)M$. But $rad(N : M)M \subseteq \langle E(N) \rangle \subseteq radN$, hence M (s.t.r.f) and so (s.t.s.r.f). Then $\langle E(N) \rangle = S - radN = radN$ for every proper submodule N of M. But by [6, Proposition 4.1], $S - radN = N$ and therefore $radN = N$.

Corollary 2.17. *Let R and M and IM be as in Lemma 2.12. Then $IM = (radI)M$.*

PROOF. Let M be a multiplication R-module and IM be an arbitrary non-zero proper submodule of M for some ideal I of R. By Lemma 2.12, $rad(IM) = (radI)M$ and by Lemma 2.16, $rad(IM) = IM$. Therefore $IM = (radI)M$.

Theorem 2.18. *Let R be a ring and let M be a multiplication R-module. Then N is a primary submodule of M if and only if it is a prime submodule of M.*

PROOF. $\Longleftarrow$. It is clear.
$\Longrightarrow$. Let M be a multiplication R-module and let N be an arbitrary primary submodule of M. Then by [2, Corollary 2], there exists a primary ideal q ($radq = p$) of R such that $N = qM$.
But by Lemma 2.16 and Corollary 2.17, $qM = (radq)M = pM$. Therefore the proof is now completed.

Corollary 2.19. *Let R be a ring which satisfies ascending chain condition on semiprime ideals and let M be a multiplication R-module. Then M is a Noetherian R-module.*

PROOF. Let M be a multiplication R-module. Then M (s.t.r.f) and hence (s.t.s.r.f). Thus by [6, Proposition 4.1], every proper submodule of M is a semiprime submodule of M. Now, let $I_1M \subseteq I_2M \subseteq I_3M \subseteq ...$ where I_i are ideals of R be ascending chain of submodules of M. Then $(I_1M : M) \subseteq (I_2M : M) \subseteq (I_3M : M) \subseteq ....$ But by [6, Proposition 2.3(ii)], $(N : M)$ is a semiprime ideal of R for any semiprime submodule N of M, hence by assumption there exists $n \in \mathbb{N}$ such that $(I_nM : M) = (I_{n+k}M : M)$ for each $k \in \mathbb{N}$. But then $(I_nM : M)M = (I_{n+k}M : M)M$ and so $I_nM = I_{n+k}M$. Therefore M is a Noetherian R-module.

It should be noted that the above results (Lemma 2.16, Corollary 2.17, Theorem 2.18, Corollary 2.19) they are not necessarily true if $M = R$, the ring itself. Because according to [6, Theorem 4.4], R (s.t.s.r.f) if we have one of the following.
(i) For every free R-module F, F (s.t.s.r.f).
(ii) For every faithful R-module B, B (s.t.s.r.f).

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REFERENCES

[1] Barnard, A., "Multiplication modules", *J. Algebra*, **71:1**(1981), 174–178.

[2] Ebrahimi Atani, S., and Callialp, F., and Tekir, $\ddot{U}$., "A short note on the primary submodules of multiplication modules", *Int. J. Algebra*,**1:8**(2007), 381–384.

[3] EL-Bast, Z.A., and Smith, P.F., "Multiplication modules", *Comm. Algebra*, **16:4**(1988), 755-779.

[4] Jenkins, J., and Smith, P.F., "On the prime radical of module over a commutative ring", *Comm. Algebra*, **20:12** (1992), 3593–3602.

[5] Lu, C.P., "Prime submodules of modules", *Comment. Math. Univ. Sancti Pauli*, **33:1**(1984), 61–96.

[6] Tavallaee, H.A., "Modules satisfying the semi-radical formula", *Hadronic Journal*, **32**(2009), 407–424.

[7] Tavallaee, H.A. and Ghalandarzadeh, S., "Semiprime submodules and envelope of modules",*IUST-International Journal of Engineering Sciences*, **14:4**(2003), 131–142.