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Dynamics of a Fractional Order Eco-Epidemiological Model

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ABSTRACT

This article discusses a fractional order eco-epidemiological model. The aim of considering the fractional order is to describe effect of time memory in the growth rate of the three populations. We investigate analytically the dynamical behavior of the model and then simulating using the Grünwald-Letnikov approximation to support our analytical results. It is found that the model has five equilibrium points, namely the origin, the survival of susceptible prey, the predator-free equilibrium, the free of infected prey equilibrium and the interior equilibrium. Numerical simulations show that the order of fractional derivative affects the behavior of solutions.

Keywords: *Fractional order, eco-epidemiological model, Grünwald-Letnikov, behavior solutions*

INTRODUCTION

The ability to support growing complexity of population rate is addressed practically and conceptually through the use of mathematical models [1]. One of the most common is the model describing the relationship between two populations, predator and prey interaction [2]. Transmission of infectious diseases involved in a prey-predator relation. A parasite population can restrict the growth rate in host mortality so the parasite can be formulated as a condition threshold for the basic reproductive rate. The presence of the disease makes the infected individuals are more easily caught by predator [3, 4]. Dynamical systems in the effect of disease in ecological systems known as eco-epidemiology.

Chattopadhyay and Arino [5] considered a three species eco-epidemiological system, namely, susceptible prey, infected prey, and predator. Prey population may be infected by a disease and the disease cannot be recovered. The predator mainly eats the infected prey where the predation follows the Holling-Type II functional response.

Saifuddin *et al.*, [6] modified Chattopadhyay and Arino model by assuming that the infected prey cannot reproduce, but both susceptible and infected are competing for the same resources. Here, the predator consumes both prey at the same rate. Later, Saifuddin *et al.*, [7] has improved the previous model by assuming the infected and susceptible prey population do not have the same competitive ability in the presence of disease in prey population. They have considered different competition coefficients for two possible interactions (intra-class competition in susceptible prey and inter-class competition between susceptible prey and infected prey). The predator population predates susceptible prey and infected prey with Holling-type II functional response.

Fractional calculus is the theory of differential and integral operators of non-integer order, and in particular to differential equations containing such operators [8]. Fractional calculus was first proposed by Leibniz and ‘Hospital in 1965 [9]. Modeling of such systems by fractional-order differential equations has the effects of the existence of time memory or long-range space interaction [10, 11]. Rivero [12] explained the order of fractional derivation is an excellent controller trajectory approach to or away from the critical point.

In recent years, the number of applications of fractional differential equations rapidly grows. Ghaziani *et al.* [13] analyzed the predator-prey population with Leslie Gower Holling type-II functional response with a fractional-order model. The results of the analysis show the increasingly complex dynamics behavior by varying

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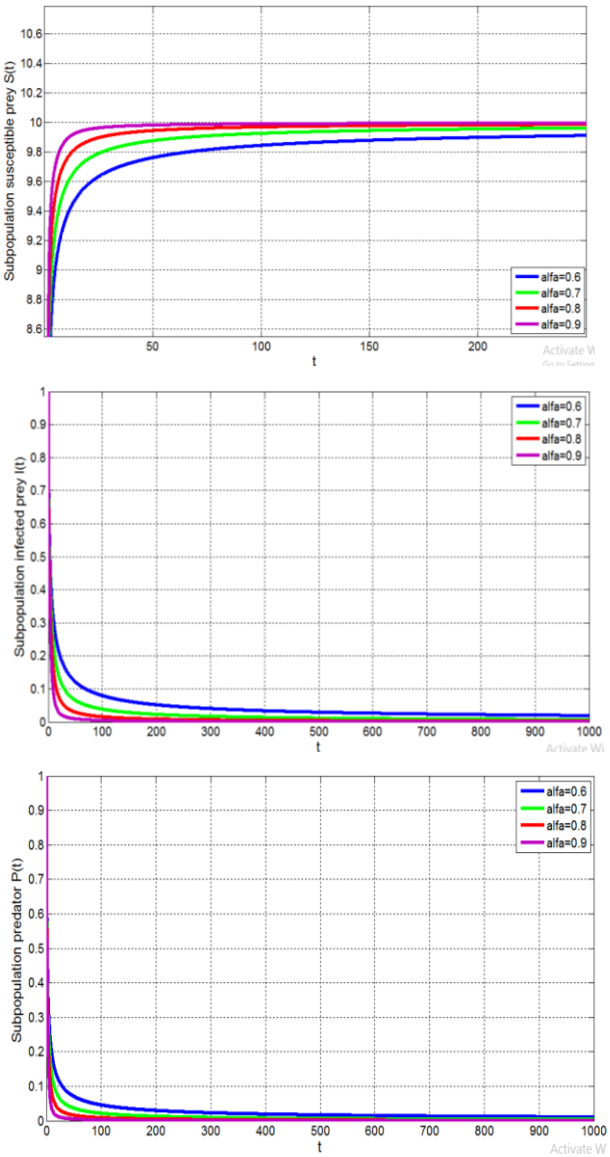

Figure 1. Numerical solution of system (1) for equilibrium point E_1 with different values: α = 0.6, α = 0.7, α = 0.8 and α = 0.9

the order of fractional while Rida *et al.* [14] modified the eco-epidemiology model as a fractional eco-epidemiological model. Model assumed the population of predators and prey populations affected by disease and predation is assumed to follow Holling-type I functional response.

In this article, we modify the eco-epidemiological model proposed by Saifuddin *et al.* [7]. It is assumed that the growth rate of the three populations not only depend on the current conditions but also take into consideration all previous state. Therefore, by changing the first order derivative into a fractional order derivative, the model proposed by Saifuddin *et al.* [7] will be modified into a fractional order eco-epidemiological model.

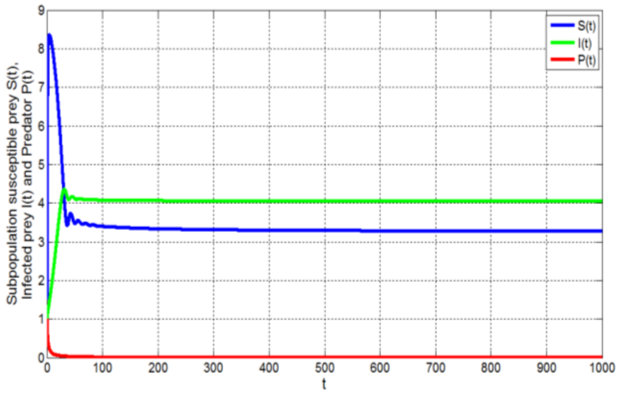

Figure 2. Numerical solution of system (1) for equilibrium point with α = 0.74

The paper investigates the dynamics of the obtained fractional order eco-epidemiological model. We present the existence of equilibrium points and investigate the local asymptotic stability. The model is simulated using the Grünwald-Letnikov approximation to support analytical results.

MATERIALS AND METHODS

Construction model

At this stage, the eco-epidemiology model proposed in [7] is modified into a fractional order eco-epidemiology model.

Determination of the equilibrium point

The equilibrium point of the system is obtained when the growth rate of populations is unchanged or zero.

Stability analysis of the equilibrium point

The local stability of the equilibrium point can be determined by the absolute value of the argument eigenvalues of the Jacobian matrix equilibrium. The condition stability shows whether the equilibrium point is stable or not. If the equilibrium point is stable, then any solution of the system with different initial values will be convergent to it, and vice versa.

Numerical experiments

Numerical simulations will be performed to support analytical results of the system behavior. Numerical solutions of the system can show the population densities with consideration all previous state. To determine the numerical solutions of a fractional order eco-epidemiological model, we use the Grünwald-Letnikov approximation method.

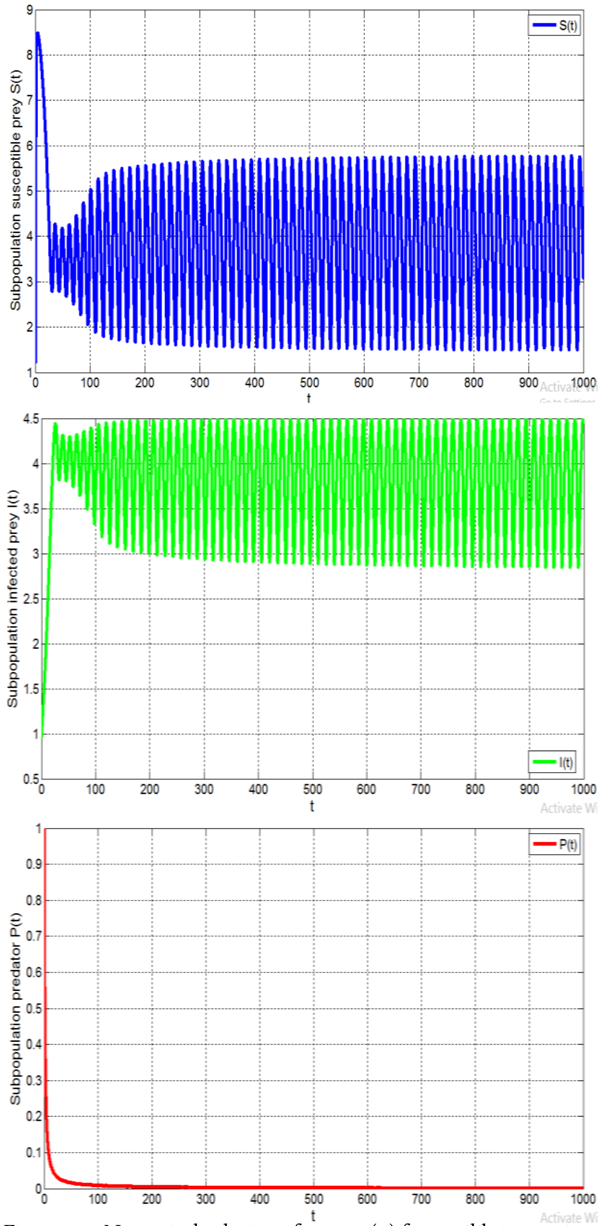

Figure 3. Numerical solution of system (1) for equilibrium point with α = 0.84

RESULTS AND DISCUSSION

Mathematical model

A fractional order eco-epidemiological model introduces by changing the first derivative into some fractional order Caputo-type derivatives. Suppose (S) is the susceptible prey population, (I) is the infected prey population and (P) is the predator population. We assume the three population densities depend on all previous state.

Susceptible prey dynamic moves exponentially towards carrying capacity, while carrying capacity is the ratio of the growth rate (r) with intra-class competition

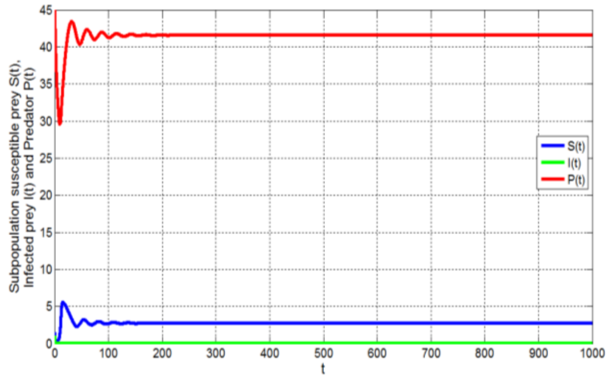

Figure 4. Numerical solution of system (1) for equilibrium point E_3 with α = 0.74

in susceptible prey (b). Saifuddin *et al.* [7] assume, the infected prey is not in a state to produce but to compete with susceptible for the same resources. Interclass competetion between susceptible prey and infected prey (c) reduces the population growth rate. Contact between susceptible prey and infected prey causes susceptible prey become infected, where β is the rate of infection of the disease, and a is the half saturation constant for the disease transmission. Infected prey density is reduced due to the death of infected prey. The death rate of infected prey is assumed to be μ.

Predator population consume both susceptible and infected prey with Holling type II functional response. Assume that α_1 is the attack rate of predator on susceptible prey, α_2 is the attack rate of predator on infected prey, c_1 and c_2 are the conversion efficiency of predator, e is the half-saturation constant of predator for susceptible prey, d is the half-saturation constant of predator for infected prey and m is the death rate of predator. Then we obtain the following fractional order system:

$$\frac{d^\alpha S}{dt^\alpha} = (r - bS) - cIS - \frac{\beta IS}{a+S} - \frac{\alpha_1 PS}{e+S}$$

$$\frac{d^\alpha I}{dt^\alpha} = \frac{\beta SI}{a+S} - \frac{\alpha_2 PS}{d+I} - \mu I,$$

$$\frac{d^\alpha P}{dt^\alpha} = \frac{c_1\alpha_1 SP}{e+S} - \frac{c_2\alpha_2 IP}{d+I} - mP, \tag{1}$$

Equilibrium point

The equilibria of the system (1) are solutions to the system:

$$\frac{d^\alpha S}{dt^\alpha} = \frac{d^\alpha I}{dt^\alpha} = \frac{d^\alpha P}{dt^\alpha} = 0$$

System has five equilibria that are:

1. $E_0 = (0,0,0)$ as the origin,
2. $E_2 = \left(\frac{r}{b}, 0, 0\right)$, as the survival of susceptible prey,
3. $E_2 = \left(\frac{\mu a}{\beta-\mu}, \frac{a[r(\beta-\mu)-b\mu a]}{(\beta-\mu)(ca+\beta-\mu)}, 0\right)$, as the survival of both susceptible and infected prey,
4. $E_3 = \left(\frac{me}{c_1\alpha_1-m}, 0, \frac{c_1e(c_1\alpha_1r-rm-bme}{(c_1\alpha_1-m)^2},\right)$, as the infected prey free equilibria,
5. $E_0 = (S_4, I_4, P_4)$, as the interior equilibria with

$$I_4 = \frac{d(m(e+S)-c_1\alpha_1 S}{[(c_2\alpha_2-m)(e+S)+c_1\alpha_1-S]},$$

$$P_4 = \left(\frac{\beta S}{a+S} - \mu\right)\left(\frac{c_2d(e+S)}{[(c_2\alpha_2-m)(e+S)+c_1\alpha_1-S]}\right),$$

$$h(S_4) = {S_4}^3 + \frac{B}{A}{S_4}^2 + \frac{C}{A}S_4 + \frac{D}{A} = 0$$

$A = (c_2\alpha_2 - m + c_1\alpha_1)b,$
$B = -[(r - ab + cd)c_1\alpha_1 - be(c_2\alpha_2 - m)],$
$B = -[(re - abe + ra)(c_2\alpha_2 - m) + rac_1\alpha_1 + (acd + \beta d)c_1\alpha_1 + d\alpha_1c_2(\mu - \beta) - md(ce - ac - \beta)],$

Stability of equilibrium point

The dynamical behavior of system (1) around each equilibrium point can be studied by investigating the local stability of the equilibrium point. The local stability is determined by the eigenvalues (λ) of the Jacobian matrix evaluated at the equilibrium point. The equilibrium point is stable if all eigenvalues satisfy $|\arg(\lambda)| > \alpha\,\pi/2$. It is found that equilibrium $E_0 = (0, 0, 0)$ is unstable node, which implies that all population will never go to extinct. Other equilibrium points are conditionally stable. The stability conditions of equilibrium points for system (1) are summarized in appendix.

Numerical simulations

In this section, some numerical simulations are given to illustrate the analysis results and show the effects of fractional order of the system. We run four simulations.

The parameters used in the first simulation are $r = 3$, $b = 0.3$, $c = 0.1$, $a = 0.5$, $\beta = 1.5$, $\alpha_1 = 0.3$, $e = 3$, $\alpha_2 = 0.1$, $d = 3$, $\mu = 1.8$, $c_1 = 0.8$, and $m = 0.8$. Based on these parameters, it is found that there are two equilibrium points namely $E_0(0,0,0)$ and $E_1(0,0,0)$. It is noted that $\frac{\beta r}{ab+r} = 1.43 < \mu = 1.8$ and $\frac{c_1\alpha_1 S_2}{eb+r} = 0.16 < m = 0.8$. According Table 1, E_1 is asymptotically stable irrespective of α while E_0 is unstable.

The numerical simulation of this case with $\alpha = 0.6$, $\alpha = 0.7$, $\alpha = 0.8$ and $\alpha = 0.9$ can be seen in Figure 1. It is shown that increasing the value of α speeds up the convergence of solutions to the equilibrium point E_1.

In this simulation, the relatively high values of death rate of infected prey and predator causes the extinction of infected prey and predator. Furthermore, the survival of susceptible prey is supported by the environment with carrying capacity (r / b).

Secondly, we set the same parameter as before except for $\mu = 1.3$. We obtain three positive equilibrium points, namely $E_0 = (0,0,0)$, $E_1 = (10,0,0)$, and $E_2 = (3.25,4.05,0)$. Based on these parameters, it is found that $\frac{c_1\alpha_1 S_2}{e+S_2} + \frac{c_2\alpha_2 I_2}{e+I_2} = 0.16 < m = 0.8$ and $r + \frac{\beta S_2}{a+S_2} = 4.3 > \mu + 2bS_2 + cI_2 + \frac{a\beta I_2}{(a+S_2)^2} = 3.87$. Furthermore, we have $|\arg(\lambda)| = 1.2$ and $\alpha^* = 0.76$. According to the condition in Table 1, if we take $\alpha = 0.74$ then $|\arg(\lambda)| = 1.2 > \frac{\alpha\pi}{2} = 1.16$, which satisfies condition (ii), therefore E_2 is asymptotically stable. Figure 2 shows that the solution of system (1) is convergent to E_2. It shows that the system goes to the co-existence of susceptible prey and infected prey, while the predator goes to extinction.

However, as seen in Figure 3, if we take $\alpha = 0.84$ then $|\arg(\lambda)| = 1.2 < \alpha\pi/2 = 1.32$. Hence, the equilibrium point E_2 is unstable. In this case, the system shows a periodic behavior. Biologically, if the predation rate is smaller than the death rate of predator, then the predator population cannot survive. On the other hand, the relatively small death rate of infected prey can cause the infected prey densities increased asymptotically. Figure 2 and 3 show that the variation of α causes changing the population densities. If $\alpha < \alpha^*$ then the population densities are convergent to E_2 while if $\alpha > \alpha^*$ then the population densities become fluctuation.

For the third simulation, we consider the same parameter values as in the second simulation, except for $m = 0.1$ We have four positive equilibrium points, namely $E_0(0,0,0)$, $E_1(10,0,0)$, $E_2(3.25,4.05,0)$, and $E_3(2.73,0,41.65)$. In this case we have $\frac{\beta S_3}{a+S_3} = 0.7 < \frac{\alpha_2 P_3}{d} + \mu = 2.6$ and $r = 3 > 2bS_3 + \frac{\alpha_2 e P_3}{(a+S_3)^2} = 2.8$. We also have $|\arg(\lambda)| = 1.23$ and $\alpha^* = 0.788$. For $\alpha = 0.74$, we have $|\arg(\lambda)| = 1.24 > \alpha\pi/2 = 1.16$. Table 1 in appendix shows that E_3 is locally asymptotically stable. Such behavior is confirmed by our simulation shown in Figure 4. This figure shows that all solutions of system (1) will converge to the equilibrium point $E_3(2.73,0,41.65)$.

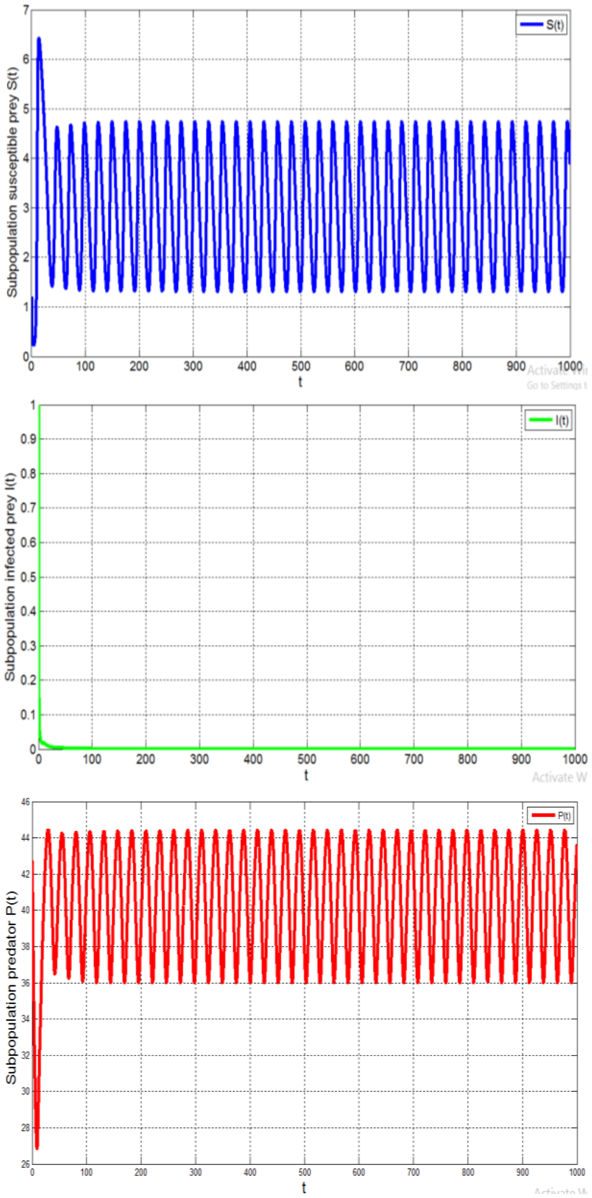

Figure 5. Numerical solution of system P(t) for equilibrium point E_3 with α = 0.85

Figure 5 illustrates solution for the case of α = 0.84, we have |arg(λ)| = 1.24 > απ/2 = 1.32. Consequently, equilibria is unstable because its stability condition is not satisfied. This behavior is depicted in Figure 5 which shows that both susceptible prey and predator populations exhibit periodic oscillation and do not converge to equilibrium point E_3(2.73,0,41.65). From Figure 4 and 5 we conclude that if α < α* then E_3 is locally asymptotically stable and vice versa.

From the ecological point, a relatively low infection rate causes the disease-free population. Furthermore, a relatively small death rate of predator will support the survival of predator population.

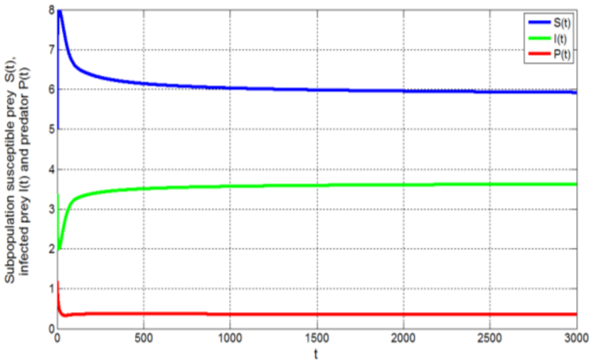

Figure 6. Numerical solution of system (1) for equilibrium point E_4 with α = 0.65

Finally, we perform simulation using the same parameter as in Simulation 1, except α2 = 1.5 and μ = 1.3. We have four positive equilibrium points, E_0(0,0,0), E_1(10,0,0), E_2(3.25,4.05,0), and E_3(5.8,3.69,0.37). Here we have b_1 = 0.88, b_2 = 0.12, b_3 = 0.01, and $D(P)$ = -0.02 < 0, then with α = 0.65, condition (ii) is satisfied and E_4 locally asymptotically stable. The numerical simulation in Figure 6 shows that solution of system (1) with initial condition S(0) = 5, I(0) = 3, and P(0) = 1 and is convergent to (5.8,3.69,0.37).

The increased attack rate on infected prey (α_2) reduces the infected prey densities and increases the predator population.

CONCLUSION

In this paper, we have studied a fractional order eco-epidemiological model. It is found that the model has five equilibrium points, i.e., the origin, the survival of susceptible prey, the predator free equilibria, the infected prey free equilibria and the interior equilibria. Numerical results show the same result with analysis. It has been found E_0 is always unstable, and the others are locally asymptotically stable under some conditions. Numerical simulations indicate fractional order α is a factor which affects the behavior of solutions. There exists α* > 0 such that if α ϵ [0, α*) the equilibrium point is asymptotically stable and the population densities become stationary. If α* < α, then the equilibrium point becomes unstable and both prey and predator population show a fluctuation behavior.

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Appendix I

Table 1. The Equilibrium Point Condition Stability

Equilibrium Point	*Condition Stability*
1. $E_0=(0,0,0)$,	Unstable
2. $E_1=\left(\frac{r}{b},0,0\right)$,	Locally asymptotically stable if only if $\frac{\beta r}{ab+r}<\mu$ and $\frac{c_1\alpha_1 r}{eb+r}<m$
3. $E_2=(S_2,I_2,0)$, with $S_2=\frac{\mu a}{\beta-\mu}$ and $I_2=\frac{a(r(\beta-\mu)-b\mu a)}{(\beta-\mu)(ca+\beta-\mu)}$	Locally asymptotically stable if only if $\frac{c_1\alpha_1 S_2}{e+S_2}+\frac{c_2\alpha_2 I_2}{d+I_2}<m$ and one of the following mutually condition holds: 1. $r+\frac{\beta S_2}{a+S_2}<\mu+2bS_2+cI_2+\frac{a\beta I_2}{(a+S_2)^2}$, (i) $\left(r-2bS_2-cI_2-\frac{a\beta I_2}{(a+S_2)^2}+\frac{\beta S_2}{a+S_2}-\mu\right)^2 \geq 4\left((r-2bS_2-cI_2)\left(\frac{\beta S_2}{a+S_2}-\mu\right)+\frac{a\beta I_2(cS_2+\mu)}{(a+S_2)^2}\right)$, (ii) $\Delta_1<0$ with $\Delta_1=\left(r-2bS_2-cI_2-\frac{a\beta I_2}{(a+S_2)^2}+\frac{\beta S_2}{a+S_2}-\mu\right)^2 -4\left((r-2bS_2-cI_2)\left(\frac{\beta S_2}{a+S_2}-\mu\right)+\frac{a\beta I_2(cS_2+\mu)}{(a+S_2)^2}\right)$, $\arg(\lambda_n)>\frac{\alpha\pi}{2}, n=2,3.$ 2. $r+\frac{\beta S_2}{a+S_2}>\mu+2bS_2+cI_2+\frac{a\beta I_2}{(a+S_2)^2}$, and $\arg(\lambda_n)>\frac{\alpha\pi}{2}, n=2,3.$
4. $E_3=(S_3,0,P_3)$, with $S_3=\frac{me}{c_1\alpha_1-m}$ and $P_3=\frac{c_1e(c_1\alpha_1 r-rm-bme)}{(c_1\alpha_1-m)^2}$.	Locally asymptotically stable if only if $\frac{\beta S_3}{a+S_3}<\frac{\alpha_2 P_3}{d}+\mu$ and one of the following mutually condition holds: 1. $r<2bS_3+\frac{\alpha_1 eP_3}{(e+S_3)^2}$, (i) $\left(r-2bS_3-\frac{\alpha_1 eP_3}{(e+S_3)^2}\right)^2\geq 4\frac{c_1\alpha_1{}^2 eS_3P_3}{(e+S_3)^3}$, (ii) $\left(r-2bS_3-\frac{\alpha_1 eP_3}{(e+S_3)^2}\right)^2-4\frac{c_1\alpha_1{}^2 eS_3P_3}{(e+S_3)^3}<0$, and $\lvert\arg(\lambda_n)\rvert>\frac{\alpha\pi}{2}, n=2,3$ 2. $r>2bS_3+\frac{\alpha_1 eP_3}{(e+S_3)^2}$ and $\lvert\arg(\lambda_n)\rvert>\frac{\alpha\pi}{2}, n=2,3$.

Equilibrium Point	*Condition Stability*
5. $E_4 = (S_4, I_4, P_4)$, with $I_4 = \frac{d(m(e+S)-c_1\alpha_1 S)}{[(c_2\alpha_2 - m)(e+S)+c_1\alpha_1 S]}$, $P_4 = \left(\frac{\beta S}{a+S} - \mu\right)\left(\frac{c_2 d(e+S)}{[(c_2\alpha_2 - m)(e+S)+c_1\alpha_1 S]}\right)$, $h(S_4) = S_4^{\,3} + \frac{B}{A}S_4^{\,2} + \frac{C}{A}S_4 + \frac{D}{A} = 0$,	Locally asymptotically stable if only if (i) $D(P) > 0, b_1 > 0, b_3 > 0, b_1 b_2 > b_3$, (ii) $D(P) < 0, b_1 \geq 0, b_2 \geq 0, b_3 > 0, \alpha < \frac{2}{3}$, $D(P) < 0, b_1 < 0, b_2 < 0, \alpha > \frac{2}{3}$, (ii) $D(P) < 0,\ b_1 > 0,\ b_2 > 0,\ b_3 > 0, b_1 b_2 = b_3, \alpha[0,1)$. $D(P) = 18 b_1 b_2 b_3 + (b_1 b_2)^2 - 4 b_3 b_1^{\,3} - 4 b_2^{\,3} - 27 b_3^{\,2}$ $b_1 = -(a_{11} + a_{22} + a_{33})$, $b_2 = a_{33}a_{22} + a_{11}a_{33} + a_{11}a_{22} - a_{13}a_{31} - a_{12}a_{21} - a_{23}a_{32}$, $b_3 = a_{13}a_{31}a_{22} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}$ $- a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} - a_{11}a_{22}a_{33}$,