



Bounds for the Laplacian spectral radius of graphs

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Abstract

This paper is a survey on the upper and lower bounds for the largest eigenvalue of the Laplacian matrix, known as the Laplacian spectral radius, of a graph. The bounds are given as functions of graph parameters like the number of vertices, the number of edges, degree sequence, average 2-degrees, diameter, covering number, domination number, independence number and other parameters.

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1. Introduction

All graphs in this paper are finite and simple. We refer to [2] for the unexplained graph theoretic terminology used here. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. Let $A(G)$ denote the $(0, 1)$ -adjacency matrix and $D(G)$ denote the diagonal matrix of the vertex degrees of G . Then the *Laplacian matrix* $L(G)$ of G is defined by $L(G) := D(G) - A(G)$. Another way to define the Laplacian matrix is the following. Fix an orientation of the edges of G , that is, for each $e_i \in E$, choose one of its end vertices as the initial vertex and the other end vertex as the terminal vertex. The *oriented vertex-edge incidence matrix*

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of G is the $n \times m$ matrix $B(G) = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1 & \text{if } v_i \text{ is the terminal vertex of } e_j \\ 0 & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $B(G)B(G)^T$ is independent of the orientation given to the edges of G and $L(G) = B(G)B(G)^T$. Kirchhoff proved a result, known as the “matrix-tree theorem”, that relates the Laplacian matrix of a graph with the number of spanning trees in it. Since the 1970s several authors from different disciplines have studied Laplacian matrices of graphs. The study of Laplacian spectrum and its relation with the structural properties of graphs has been one of the most attracting features of the subject.

Clearly, $L(G)$ is a real, symmetric and positive semi-definite matrix. So all its eigenvalues are real and non-negative. Since the sum of the entries in each row of $L(G)$ is zero, the all one vector $\mathbf{e} = [1, \dots, 1]^T$ is an eigenvector of $L(G)$ corresponding to the smallest eigenvalue 0, in particular, $L(G)$ is singular. We refer the reader to [13, 37, 33, 34] and the references therein for more on the Laplacian matrix and its eigenvalues. The largest eigenvalue of $L(G)$ is called the *Laplacian spectral radius* of G and we denote it by $\lambda(G)$. The Laplacian matrix $L(G)$ of G depends on the ordering of its vertices. However, Laplacian matrices afforded by different vertex orderings of the same graph are permutation similar. So two isomorphic graphs have the same Laplacian spectrum.

The Laplacian matrix $L(G)$ of G is irreducible if and only if G is connected. If G is disconnected, then $L(G)$ is similar to a block diagonal matrix, where each block is the Laplacian matrix of some connected component of G . So, if G_1, G_2, \dots, G_k are the connected components of G , then $\lambda(G) = \max\{\lambda(G_i) : 1 \leq i \leq k\}$. Let W be the set of all unit vectors in \mathbb{R}^n , that is, $W = \{X \in \mathbb{R}^n \mid X^T X = 1\}$. By Rayleigh-Ritz theorem [21, p.176], we have

$$\lambda(G) = \max_{X \in W} X^T L(G) X.$$

If X is a unit eigenvector of $L(G)$ corresponding to $\lambda(G)$, then

$$\lambda(G) = X^T L(G) X = \sum_{v_i v_j \in E} (x_i - x_j)^2,$$

where $X^T = [x_1, x_2, \dots, x_n]$. Here for two distinct adjacent vertices v_i and v_j , $v_i v_j \in E$ denotes the corresponding edge.

The second smallest eigenvalue of $L(G)$, denoted by $a(G)$, is called the *algebraic connectivity* of G by Fiedler [11] and has received a good deal of attention so far. From the Perron-Frobenius theorem applied to the matrix $(n-1)I - L(G)$, it follows that $a(G)$ is positive if and only if G is connected. Fiedler proved that $\lambda(G) = n - a(\overline{G})$, where \overline{G} denotes the complement graph of G [11, 3.7(1°)]. Thus information on the Laplacian spectral radius of a graph can be obtained from the algebraic connectivity of the complement graph of it.

Several researchers have extensively studied the Laplacian spectral radius of graphs and obtained various bounds for it with respect to varying graph parameters and whenever possible,

the corresponding extremal graphs have been characterized. Information on the Laplacian spectral radius of a graph is also useful in several other areas like: combinatorial optimization (see [38, 39, 42]), communication network (see [48]), theoretical chemistry (see [18, 19]) etc.

Starting from the very first result, this paper is a survey on the upper and lower bounds of the Laplacian spectral radius as a function of graph parameters like the number of vertices, the number of edges, degree sequence, average 2-degrees, diameter, matching number, covering number, domination number, independence number etc. We have organized the paper as follows: Sections 2 and 3 are dedicated to the upper bounds, and Section 4 is for the lower bounds.

Along with other results, the following are the three basic lemmas which are frequently used in the proof of many of the bounds. Recall that if A is a nonnegative square matrix, then the spectral radius of A is an eigenvalue of A [36, Theorem 4.2, p.14]. The following lemma says that the largest eigenvalue of A is bounded by the minimal and maximal row sums of A [36, Theorem 1.1, p.24].

Lemma 1.1. [36] *Let A be a $k \times k$ nonnegative matrix with maximal eigenvalue η and row sums r_1, r_2, \dots, r_k . Then*

$$r \leq \eta \leq R,$$

where $r = \min\{r_i : 1 \leq i \leq k\}$ and $R = \max\{r_i : 1 \leq i \leq k\}$. If A is irreducible, then equality can hold on either side of the above inequality if and only if all row sums of A are equal.

The matrix $Q(G) := D(G) + A(G)$ is called the *signless Laplacian matrix* of G . Let $\rho(G)$ denote the largest eigenvalue of $Q(G)$. The following relation between $\lambda(G)$ and $\rho(G)$ is proved in [60, Lemma 2.1], also see [41, Theorem 2.3]. We note that $Q(G)$ and $L(G)$ are similar for a bipartite graph G .

Lemma 1.2. [60] *Let G be a graph. Then $\lambda(G) \leq \rho(G)$. If G is a bipartite graph, then equality holds. Conversely, if G is connected and equality holds, then G is a bipartite graph.*

The *line graph* L_G of G is the graph whose vertices correspond to the edges of G , with two distinct vertices of L_G are adjacent if and only if the corresponding edges in G have a vertex in common. If G has no isolated vertex, then L_G is connected if and only if G is connected. The *vertex-edge incidence matrix* of G is the $n \times m$ matrix $R(G) = (r_{ij})$, where

$$r_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise.} \end{cases}$$

Then $R(G)R(G)^T = D(G) + A(G) = Q(G)$ and $R(G)^T R(G) = 2I_m + A(L_G)$. So, the largest eigenvalue $\rho(G)$ of $Q(G)$ is equal to $2 + \mu(L_G)$, where $\mu(L_G)$ denotes the largest eigenvalue of the adjacency matrix $A(L_G)$ of L_G . Hence Lemma 1.2 implies the following (also see [47, Lemma 2]).

Lemma 1.3. *Let G be a graph and L_G be its line graph. If $\mu(L_G)$ is the largest eigenvalue of $A(L_G)$, then $\lambda(G) \leq 2 + \mu(L_G)$. If G is a bipartite graph, then equality holds. Conversely, if G is connected and equality holds, then G is a bipartite graph.*

We observe that many results available in the literature on the bounds of the Laplacian spectral radius of a graph are stated stipulating ‘connectedness’ of the graph a priori. However, most of the given proofs could work well for disconnected graphs also and if necessary, with assumptions like the graph has at least one edge or has no isolated vertex. We find that while listing the above lemmas as preliminary results, connectedness of the graph is assumed in Lemmas 1.2, 1.3, and irreducibility of the matrix is assumed in Lemma 1.1 from the beginning itself, which forces to state the new bounds for connected graphs only. Confinement to connected graphs only simplifies the study of the equality case of a given bound, though equality may hold good for some less obvious disconnected graphs as well (for example, see the equality case of the bound (20) in the next section).

We have tried here to state many of the bounds without restricting to connected graphs only, unless it is necessary. However, connectedness is generally assumed to characterize the equality cases. Note that if G is a graph with an isolated vertex v , then $\lambda(G) = \lambda(G \setminus \{v\})$. Therefore, we assume throughout that *all graphs are without any isolated vertices*.

2. Upper Bounds

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E . For each vertex $v_i \in V$, we denote by N_i the neighborhood of v_i , that is, the set of vertices of G adjacent to v_i , and by $d_i = d(v_i)$ the *degree* of the vertex v_i (we may write $d_G(v_i)$ if more than one graph is under consideration). So $d_i = |N_i|$. The minimal and maximal vertex degrees of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. For each $1 \leq i \leq n$, we denote by m_i the average of the degrees of the vertices adjacent to v_i , that is,

$$m_i = \frac{1}{d_i} \left(\sum_{v_j \in N_i} d_j \right).$$

The integer m_i is called the *average 2-degree* of the vertex v_i . The *degree sequence* of G is the non-increasing sequence of its vertex degrees. Whenever necessary, the vertices of G can be renumbered so that $d_i \geq d_{i+1}$ for $1 \leq i \leq n-1$. In that case, we say that G has degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$. Note that $\delta = d_n \geq 1$, since we are considering graphs without isolated vertices.

A bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$ is said to be *semiregular* if the vertices in each V_i have the same degree for $i = 1, 2$. Here regular bipartite graphs are also considered to be semiregular.

The first two upper bounds for the Laplacian spectral radius of a graph were given by Fiedler in 1973 in terms of the number of vertices and the maximal vertex degree.

Theorem 2.1. [11] *Let G be a graph with n vertices and maximal vertex degree Δ . Then*

$$\lambda(G) \leq n; \tag{1}$$

and

$$\lambda(G) \leq 2\Delta. \tag{2}$$

Equality holds in (1) if and only if \overline{G} is disconnected. If G is connected, then equality holds in (2) if and only if G is a regular bipartite graph.

The bound (1) follows from the relation $\lambda(G) = n - a(\overline{G})$, also see [1, Theorem 1]. The bound (2) was proved in [11, 3.7(5°)]. The following bound (3) by Anderson and Morley in [1, Theorem 2] is an improvement of (2).

Theorem 2.2. [1] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \max \{d_i + d_j : v_i v_j \in E\}. \quad (3)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

The equality case of (2) for connected graphs is a consequence of Theorem 2.2. The following bound (4) was given by Li and Zhang [25, Theorem 3.2] in terms of the largest three vertex degrees.

Theorem 2.3. [25] Let $G = (V, E)$ be a graph with degree sequence $d_1 \geq d_2 \geq \dots \geq d_n$. Then

$$\lambda(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)}. \quad (4)$$

If G is connected, then equality holds if and only if G is either a regular bipartite graph or, a path with three or four vertices.

The bounds (3) and (4) are not comparable in general, see the examples given in [35, p.34]. However, the following bound (5) mentioned in [25, Remark 1] is better than both (3) and (4). A different proof of (5) was given by Pan in [41, Theorem 2.9], where the connected graphs achieving this bound were determined.

Theorem 2.4. [25, 41] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Let $a = \max \{d_i + d_j \mid v_i v_j \in E\}$ and let $v_k v_h \in E$ be such that $a = d_k + d_h$. Define $b = \max \{d_i + d_j \mid v_i v_j \in E \setminus \{v_k v_h\}\}$. Then

$$\lambda(G) \leq 2 + \sqrt{(a - 2)(b - 2)}. \quad (5)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph or a path with four vertices.

The following bound given by Zhang and Li [59, Corollary 4.4] is a further improvement of (5).

Theorem 2.5. [59] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq 2 + \max \left\{ \sqrt{(d_i + d_j - 2)(d_i + d_k - 2)} \right\}, \quad (6)$$

where the maximum is taken over all edges $v_i v_j, v_i v_k \in E$ with $v_j \neq v_k$. Further, if G is connected, then equality holds if and only if G is a semiregular bipartite graph or a path with four vertices.

The following bound (7) was given by Merris in [35, p.34] which improves (3). An alternative proof of (7) was given by Pan in [41, Theorem 2.4], where the extremal connected graphs were also characterized.

Theorem 2.6. [35, 41] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(G) \leq \max \{d_i + m_i \mid v_i \in V\}. \quad (7)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

Merris has given examples in [35, p.34] showing that the bounds (7) and (5) are not comparable in general. The following bound (8) given by Li and Zhang in [26, Theorem 3] improves (7). Another proof of (8) was given by Pan in [41, Theorem 2.10], where necessary and sufficient conditions were provided for the equality case (also see [59, Theorem 4.5]).

Theorem 2.7. [26, 41] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(G) \leq \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} : v_i v_j \in E \right\}. \quad (8)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

The following bound (9) given by Pan in [41, Theorem 2.11] is a further improvement of (8). Though connectedness of the graph is assumed in [41, Theorem 2.11], the given proof works for any graph.

Theorem 2.8. [41] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Define

$$t = \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} : v_i v_j \in E \right\}.$$

Let $v_k v_h \in E$ be such that $t = \frac{d_k(d_k + m_k) + d_h(d_h + m_h)}{d_k + d_h}$. Define

$$s = \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} : v_i v_j \in E \setminus \{v_k v_h\} \right\}.$$

Then

$$\lambda(G) \leq 2 + \sqrt{(t-2)(s-2)}. \quad (9)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph or a path with four vertices.

For certain graphs, some of the above bounds could give results which are greater than the number of vertices and so they become trivial bounds comparing to (1). Addressing this problem, Rojo et al. gave the following bound (10) in [44, Theorem 4] whose value never exceeds the number of vertices.

Theorem 2.9. [44] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \max \{d_i + d_j - |N_i \cap N_j| : 1 \leq i \neq j \leq n\}. \quad (10)$$

This upper bound for $\lambda(G)$ is always less than or equal to n .

Note that the maximum in (10) is taken over all pairs of distinct vertices, a reason for which the bounds (3) and (10) are not comparable in general (see the example given in [5, p.271]). Restricting the maximum over pairs of adjacent vertices only, Das in [5, Theorem 2.1] gave the following bound (11) which is always better than (3) and (10). The result for equality case of (11) was conjectured by him in [7, Problem 2.17] and was proved by Yu et al. in [55, Theorem 2.2].

To state the equality case of (11), we define the following: Let $H = (V, E)$ be a semiregular bipartite graph with bipartition $V = V_1 \cup V_2$ and let $H^+ = (V, E^+)$ be the supergraph of H adding new edges by joining those pairs of vertices in V_1 (respectively, in V_2) which have the same set of neighbors in V_2 (respectively, in V_1), if such pairs exist. Define $\mathfrak{H}^+ = \{H^+ : H \text{ is a semiregular bipartite graph}\}$.

Theorem 2.10. [5, 55] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \max \{d_i + d_j - |N_i \cap N_j| : v_i v_j \in E\}. \quad (11)$$

If G is connected, then equality holds if and only if $G \in \mathfrak{H}^+$.

The following bound (12) was given by Rojo et al. in [45, Corollary 16] based on vertex degrees and the number of vertices.

Theorem 2.11. [45] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \frac{1}{n-1} \left(\sum_{i=1}^n d_i \right) + \sqrt{\frac{n-2}{n-1} f(G)}, \quad (12)$$

where $f(G) = \sum_{i=1}^n d_i(d_i + 1) - \frac{1}{n-1} \left(\sum_{i=1}^n d_i \right)^2$.

In [58, Lemma 3.2], Zhang and Li gave an upper bound for the sum of squares of the vertex degrees of a graph in terms of the number of vertices and the number of edges. Using this as a tool, the following bound (13) was obtained in [58, Theorem 3.3] in terms of the number of vertices and edges of a graph.

Theorem 2.12. [58] Let $G = (V, E)$ be a graph with n vertices and m edges. Let M be the minimum of $m^2(n-4) + 2m(n-1)$ and $2mn(n-1) - 4m^2$. Then

$$\lambda(G) \leq \frac{1}{n-1} \left(2m + \sqrt{(n-2)M} \right). \quad (13)$$

If G is connected, then equality holds if and only if G is the complete graph K_n or the star graph $K_{1,n-1}$.

In [10, Theorem 1], de Caen had given another upper bound for the sum of squares of the vertex degrees of a graph in terms of the number of vertices and the number of edges. Using de Caen's inequality as a tool, Li and Pan proved in [22, Theorem 3.1] the following bound (14) in terms of the number of vertices and edges (their proof would work for any graph) and characterized the connected graphs for which equality is achieved.

Theorem 2.13. [22] *Let $G = (V, E)$ be a graph with n vertices and m edges. Then*

$$\lambda(G) \leq \frac{2m + \sqrt{(n-2)m(n(n-1) - 2m)}}{n-1}. \quad (14)$$

If G is connected, then equality holds if and only if G is the star graph or the complete graph K_n .

For connected graphs, the bound (14) can be obtained without using de Caen's inequality, see the discussion in [4, p.1963]. It can be seen that when $m \geq n-1$, in particular when G is connected, the bound (14) is an improvement of (13). We note that, for connected graphs, the bounds (13) and (14) always give values which are greater than or equal to the number n of vertices, see [7, Theorem 2.18].

In [22, Theorems 3.2, 3.3], Li and Pan proved the following two bounds (15) and (16), also see [7, Corollaries 2.10, 2.11]. In [46, Theorem 3.1], Shi gave a different proof of (16) and then derived the bound (15) in [46, Corollary 3.1] as an application of (16). From the proof of [46, Corollary 3.1], it follows that (16) is better than (15).

Theorem 2.14. [22, 46] *Let $G = (V, E)$ be a graph with n vertices, m edges, vertex degrees d_i , average 2-degrees m_i , maximal vertex degree Δ and minimal vertex degree δ . Then*

$$\lambda(G) \leq \sqrt{2\Delta^2 + 4m + 2\Delta(\delta - 1) - 2\delta(n-1)}; \quad (15)$$

and

$$\lambda(G) \leq \max \left\{ \sqrt{2d_i(d_i + m_i)} : v_i \in V \right\}. \quad (16)$$

If G is connected, then equality holds in each of (15) and (16) if and only if G is a regular bipartite graph.

Remark 2.1. The equality case of (16) in [46, Theorem 3.1] was characterized as the following: If G is connected, then equality holds in (16) if and only if G is bipartite and the value $d_i^2 + d_i m_i$ is independent of the vertex $v_i \in V$. It follows that a bipartite graph is regular if and only if $d_i^2 + d_i m_i$ is the same for all i . A direct proof of this fact can be given following an argument similar to the proof of Proposition 2.1 below (also see Remark 2.2 after Theorem 2.25).

The following bound (17) given by Zhang and Luo in [61, Theorem 3.2] is an improvement of (15). They stated this bound for connected mixed graphs but their proof works for any mixed graph. Since the Laplacian matrix of a simple graph is consistent with the Laplacian matrix of the associated mixed graph in which all edges are oriented, so their proof remains valid for simple graphs also.

Theorem 2.15. [61] Let G be a graph with n vertices, m edges, maximal vertex degree Δ and minimal vertex degree δ . Then

$$\lambda(G) \leq \Delta + \sqrt{2m + \Delta(\delta - 1) - \delta(n - 1)}. \quad (17)$$

If G is connected, then equality holds if and only if G is a regular bipartite graph.

The following bound (18), which is an improvement of (7), was given by Zhang and Luo in [61, Theorem 3.4] more generally for mixed graphs. The same bound was obtained by Das in [7, Theorem 2.14] (his proof would work for any graph without isolated vertices).

Theorem 2.16. [7, 61] Let G be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(G) \leq \max \left\{ \frac{1}{2} \left(d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j} \right) : v_i v_j \in E \right\}. \quad (18)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph.

In [5, Theorem 2.5], Das proved the following bound (19) which improves (16). The connected graphs achieving this bound were characterized by him in [7, Theorem 2.9].

Theorem 2.17. [5, 7] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . For $1 \leq i \leq n$, define $m'_i = \frac{1}{d_i} \left(\sum_{v_j \in N_i} (d_j - |N_i \cap N_j|) \right)$. Then

$$\lambda(G) \leq \max \left\{ \sqrt{2d_i(d_i + m'_i)} : 1 \leq i \leq n \right\}. \quad (19)$$

If G is connected, then equality holds if and only if G is a regular bipartite graph.

The following bound (20) was given by Shu et al. for connected graphs based on the degree sequence [47, Theorem 1]. However, their characterization of the connected graphs for the equality case was incomplete as pointed out by K. C. Das in [3, p.283]. In [62, Theorem 3], Zhou and Cho obtained the same bound and prescribed the conditions for equality case for any graph (not necessarily connected).

Theorem 2.18. [47, 62] Let G be a graph with n vertices and degree sequence $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$. Then

$$\lambda(G) \leq d_n + \frac{1}{2} + \sqrt{\left(d_n - \frac{1}{2}\right)^2 + \sum_{i=1}^n d_i(d_i - d_n)}. \quad (20)$$

Further, equality holds if and only if G is a regular graph with at least one bipartite component, or G is the disjoint union of a star graph and (possibly) some K_2 's.

The following two bounds were given by Li and Pan in [23, Theorems 3.1, 3.2], also see [28, Theorem 4] for the bound (22).

Theorem 2.19. [28, 23] Let G be a graph with n vertices, m edges, maximal vertex degree Δ and minimal vertex degree δ . Then the following hold:

$$\lambda(G) \leq \frac{1}{2} \left(\delta - 1 + \sqrt{(\delta - 1)^2 + 8(\Delta^2 + 2m - \delta(n - 1))} \right); \quad (21)$$

$$\lambda(G) \leq \frac{1}{2} \left((\Delta + \delta - 1) + \sqrt{(\Delta + \delta - 1)^2 + 8(2m - \delta(n - 1))} \right). \quad (22)$$

If G is connected, then equality holds in both cases if and only if G is a regular bipartite graph.

The next three bounds (23) – (25) were given by Zhang in [56, Theorems 1.1, 1.2, 3.2] for connected graphs (though the given proofs work for any graph). The bound (23) is an improvement of (16). Using the inequality in [17, Lemma 2.1], it can be seen that (23) is better than (17), and (24) is better than (15). Based on the observation made by Wang in [51, Remark 1], the statement in the equality case of (23) is modified from its original one.

Theorem 2.20. [56] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then the following hold:

$$\lambda(G) \leq \max \left\{ d_i + \sqrt{d_i m_i} : v_i \in V \right\}; \quad (23)$$

$$\lambda(G) \leq \max \left\{ \sqrt{d_i(d_i + m_i) + d_j(d_j + m_j)} : v_i v_j \in E \right\}; \quad (24)$$

$$\lambda(G) \leq \max \left\{ 2 + \sqrt{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4} : v_i v_j \in E \right\}. \quad (25)$$

If G is connected, then the following hold:

- (i) Equality in (23) holds if and only if G is a regular bipartite graph.
- (ii) Equality in (24) holds if and only if G is a semiregular bipartite graph.
- (iii) Equality in (25) holds if and only if G is a semiregular bipartite graph or a path on 4 vertices.

The following upper bound (26) was given by Guo in [15, Theorem 1].

Theorem 2.21. [15] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \max \left\{ \frac{d_i + \sqrt{d_i^2 + 8d_i m'_i}}{2} : v_i \in V \right\}, \quad (26)$$

where m'_i are defined as in Theorem 2.17. Further, if G is connected, then equality holds if and only if G is a regular bipartite graph.

The following four bounds (27) – (30) were given by Das in [9, Theorems 5.1, 5.4, 5.7].

Theorem 2.22. [9] Let G be a graph with n vertices and degree sequence $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$. Then, for $d_n = 1$,

$$\lambda(G) \leq 2 + \sqrt{\sum_{i=1}^n d_i(d_i - 1) - d_1 + 1}; \quad (27)$$

and for $d_n \geq 2$,

$$\lambda(G) \leq 2 + \sqrt{\sum_{i=1}^n d_i(d_i - 1) - \left(\frac{1}{2} \sum_{i=1}^n d_i - 1\right) (2d_n - 2) + (2d_n - 3)(2d_1 - 2)}. \quad (28)$$

If G is connected, then equality holds in (27) if and only if G is a star graph, and equality holds in (28) if and only if G is a regular bipartite graph.

Theorem 2.23. [9] Let G be a graph with n vertices, m edges, maximal degree Δ , minimal degree δ and average 2-degrees m_i . Set $\theta = \max\{m_i : 1 \leq i \leq n\}$. Then the following hold:

$$\lambda(G) \leq \frac{1}{2} \left(\Delta + \sqrt{\Delta^2 + 4\theta \left[\frac{2m}{n-1} + \frac{n-2}{n-1} \Delta + (\Delta - \delta) \left(1 - \frac{\Delta}{n-1} \right) \right]} \right); \quad (29)$$

$$\lambda(G) \leq \frac{2m}{n-1} + \frac{n-2}{n-1} \Delta + (\Delta - \delta) \left(1 - \frac{\Delta}{n-1} \right). \quad (30)$$

If G is connected, then the following hold:

- (i) Equality in (29) holds if and only if G is a regular bipartite graph.
- (ii) Equality in (30) holds if and only if G is a star graph or a regular bipartite graph.

In [51, Theorems 2.6, 2.7, 2.10], the following three bounds (31) – (33) were obtained by Wang for connected graphs (though the given proofs work for any graph). An alternative proof of the bound (31) was given by Zhu in [63, Theorem 3.7].

Theorem 2.24. [51, 63] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, m edges, vertex degrees d_i and average 2-degrees m_i . Then the following hold.

$$\lambda(G) \leq \max \left\{ \frac{1}{2} \left(d_i + d_j + \sqrt{(d_i - d_j)^2 + 4\sqrt{d_i d_j m_i m_j}} \right) : v_i v_j \in E \right\}; \quad (31)$$

$$\lambda(G) \leq 2 + \max \left\{ \sqrt{\frac{(d_i + d_j - 2)(d_i^2 m_i + d_j^2 m_j - 2d_i d_j)}{d_i d_j}} : v_i v_j \in E \right\}; \quad (32)$$

$$\lambda(G) \leq 2 + \sqrt{\sum_{v_i \in V} d_i^2 - 2m - (m-1)r_0 + (r_0-1)r_1}; \quad (33)$$

where $r_0 = \min \{d_i + d_j - 2 \mid v_i v_j \in E\}$ and $r_1 = \max \{d_i + d_j - 2 \mid v_i v_j \in E\}$ in the last inequality. If G is connected, then the following hold:

- (i) Equality in (31) holds if and only if G is a semiregular bipartite graph.
- (ii) Equality in each of (32) and (33) holds if and only if G is a semiregular bipartite graph or a path with four vertices.

The following bound (34) given by Guo in [16, Theorem 2.2] is always better than the bound (23). In fact, Guo proved more general results in [16, Theorem 2.1] (for which connectedness of the graph is not necessary), as consequences of which the bounds (3), (7), (18), (23) and (34) could be obtained.

Theorem 2.25. [16] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \leq \max \left\{ d_i + \frac{1}{\sqrt{d_i}} \sum_{v_i v_j \in E} \sqrt{d_j} : v_i \in V \right\}. \quad (34)$$

If G is connected, then equality holds if and only if G is a regular bipartite graph.

Remark 2.2. The equality case of (34) in [16, Theorem 2.2] was characterized as the following: If G is connected, then equality holds in (34) if and only if G is bipartite and the value $a_i = d_i + \frac{1}{\sqrt{d_i}} \sum_{v_i v_j \in E} \sqrt{d_j}$ is independent of the vertex $v_i \in V$. We note that, for a bipartite graph, the condition that the values a_i are the same for all i is equivalent to that the graph is regular. This follows from the following proposition.

Proposition 2.1. Let $G = (V, E)$ be a bipartite graph. Define $a_y = d(y) + \frac{1}{\sqrt{d(y)}} \sum_{yw \in E} \sqrt{d(w)}$ for each vertex $y \in V$. Then G is regular if and only if the values a_y are the same for all $y \in V$.

Proof. If G is regular, then clearly a_y is independent of the vertex y . Conversely, assume that a_y is the same for all $y \in V$. We show that G is regular.

Let $V = X \cup Y$ be a bipartition of V . Let $u \in V$ be a vertex of maximal degree. Without loss of generality, we may assume that $u \in X$. Let $v \in Y$ be a vertex of minimal degree among all the vertices in Y . Then $d(u) \geq d(v)$. Now

$$a_u = d(u) + \frac{1}{\sqrt{d(u)}} \sum_{uw \in E} \sqrt{d(w)} \geq d(u) + \frac{1}{\sqrt{d(u)}} (d(u) \sqrt{d(v)}) = d(u) + \sqrt{d(u)d(v)};$$

and

$$a_v = d(v) + \frac{1}{\sqrt{d(v)}} \sum_{vw \in E} \sqrt{d(w)} \leq d(v) + \frac{1}{\sqrt{d(v)}} (d(v) \sqrt{d(u)}) = d(v) + \sqrt{d(v)d(u)}.$$

Since $a_u = a_v$, we get $d(u) + \sqrt{d(u)d(v)} \leq d(v) + \sqrt{d(v)d(u)}$ and this gives $d(u) \leq d(v)$. So $d(u) = d(v)$. It follows that all vertices in Y have the same degree and equal to $d(u)$. Now let $x \in X$. We have $d(x) \leq d(u)$ and

$$a_x = d(x) + \frac{1}{\sqrt{d(x)}} \sum_{xw \in E} \sqrt{d(w)} = d(x) + \frac{1}{\sqrt{d(x)}} (d(x) \sqrt{d(u)}) = d(x) + \sqrt{d(x)d(u)}.$$

Then

$$2d(u) = a_u = a_x = d(x) + \sqrt{d(x)d(u)} \leq d(x) + d(u),$$

which gives $d(u) \leq d(x)$. So $d(x) = d(u)$. Thus any two vertices in X have the same degree and equal to $d(u)$. Hence G is regular. \square

The following bound (35) was given by Yu in [53, Theorem 2.6] in terms of the degree sequence of the line graph of a given graph. This bound is better than (5).

Theorem 2.26. [53] Let G be a graph with m edges and let $t_1 \geq t_2 \geq \dots \geq t_m$ be the degree sequence of the line graph of G . Then

$$\lambda(G) \leq \min_{1 \leq i \leq m} \left\{ \frac{t_i + 3 + \sqrt{(t_i + 1)^2 + 4(i - 1)(t_1 - t_i)}}{2} \right\}. \quad (35)$$

If G is connected, then equality holds if and only if G is a semiregular bipartite graph or $G \simeq P_2^{k,k}$, where $P_2^{k,k}$ is the graph obtained from a path P_2 on two vertices by adjoining k vertices to each vertex of P_2 .

In [63], the following bounds (36) – (41) were given by Zhu. The bound (41) is better than (8), and (40) is better than (36).

Theorem 2.27. [63] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then the following hold.

$$\lambda(G) \leq \max \left\{ d_i \frac{m_i}{d_j} + d_j \frac{m_j}{d_i} : v_i v_j \in E \right\}; \quad (36)$$

$$\lambda(G) \leq \max \left\{ d_i \sqrt{\frac{m_i}{d_j}} + d_j \sqrt{\frac{m_j}{d_i}} : v_i v_j \in E \right\}; \quad (37)$$

$$\lambda(G) \leq \max \left\{ \frac{d_i \sqrt{d_i + m_i} + d_j \sqrt{d_j + m_j}}{\sqrt{d_i + d_j}} : v_i v_j \in E \right\}; \quad (38)$$

$$\lambda(G) \leq \max \left\{ \frac{d_i(\sqrt{d_i} + \sqrt{m_i}) + d_j(\sqrt{d_j} + \sqrt{m_j})}{\sqrt{d_i} + \sqrt{d_j}} : v_i v_j \in E \right\}; \quad (39)$$

$$\lambda(G) \leq \max \left\{ 2 + \sqrt{(T(i, j) - 2)(T(i, k) - 2)} : v_i v_j, v_i v_k \in E, v_j \neq v_k \right\}; \quad (40)$$

$$\lambda(G) \leq \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} - \frac{2}{d_i + d_j} \sum_{v_k \in N_i \cap N_j} d_k : v_i v_j \in E \right\}; \quad (41)$$

where $T(s, t) = \frac{d_s}{d_t} m_s + \frac{d_t}{d_s} m_t$ in the inequality (40).

In fact, Zhu proved three general results [63, (2.1), (2.5), (2.6)] from which the above bounds (36) – (41) are derived as consequences. The bounds (3), (5), (9) and (11) could also be obtained as consequences of these general results.

In [52, Theorems 2.7, 2.8, 2.9], the following three bounds (42) – (44) were given by Wang et al. and the connected graphs achieving these bounds were characterized. The bound (42) is an improvement of (32).

Theorem 2.28. [52] Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Define

$$a = \max \left\{ \sqrt{(d_i + d_j - 2)(d_i^2 m_i + d_j^2 m_j - 2d_i d_j)/d_i d_j} : v_i v_j \in E \right\}$$

and

$$b = \max \left\{ \sqrt{(d_i + d_j - 2)(d_i^2 m_i + d_j^2 m_j - 2d_i d_j)/d_i d_j} : v_i v_j \in E \setminus \{v_k v_h\} \right\},$$

where a is attained by some edge $v_k v_h \in E$. Then the following hold:

$$\lambda(G) \leq 2 + \sqrt{ab}; \quad (42)$$

$$\lambda(G) \leq \max \left\{ d_i + \frac{d_i(m_i + \sqrt{m_i})}{d_i + \sqrt{d_i}} : v_i \in V \right\}; \quad (43)$$

$$\lambda(G) \leq \max \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j) - 2(\Delta_i + \Delta_j)}{d_i + d_j - |N_i \cap N_j|} : v_i v_j \in E \right\}; \quad (44)$$

where Δ_k denotes the number of triangles associated with the vertex $v_k \in V$. If G is connected, then the following hold:

- (i) Equality in (42) holds if and only if G is a semiregular bipartite graph or a path with four vertices.
- (ii) Equality in (43) holds if and only if G is a regular bipartite graph.
- (iii) Equality in (44) holds if and only if G is a semiregular bipartite graph.

Recall that a *clique* in a graph G is a set of pairwise adjacent vertices. The *clique number* of G , denoted by $\omega(G)$, is the maximum size of a clique in G . The next upper bound involving clique number was given by Lu et al. in [30, Theorem 2.5].

Theorem 2.29. [30] Let G be a graph with n vertices, m edges, maximum degree Δ and clique number $\omega = \omega(G)$. Then

$$\lambda(G) \leq \Delta + \sqrt{2m \left(1 - \frac{1}{\omega} \right)}. \quad (45)$$

3. Upper Bounds for Special Classes of Graphs

In this section, we survey the results known for the upper bounds of the Laplacian spectral radius of some special classes of graphs: Trees, Non-regular graphs, Triangular graphs, Triangle-free graphs, Maximal planner graphs and Bipartite graphs.

3.1. Trees

For a tree, Stevanović gave the following strict upper bound (46) in [49, Theorem 1] in terms of the maximal vertex degree.

Theorem 3.1. [49] *Let T be a tree with maximal vertex degree Δ . Then*

$$\lambda(T) < \Delta + 2\sqrt{\Delta - 1}. \quad (46)$$

We denote by $d(u, v)$ the distance between two vertices u and v in a graph. The following strict upper bound (47) given by Rojo in [43, Theorem 3] improves the bound (46) if $\sigma_1 < \Delta$, the maximal vertex degree of the tree.

Theorem 3.2. [43] *Let T be a tree with maximal vertex degree Δ and u be a vertex of T of degree $d(u) = \Delta$. Let $k - 1$ be the maximal distance from u to any vertex of T . For $j = 1, \dots, k - 2$, define $\sigma_j = \max \{d(v) : d(u, v) = j\}$. Then*

$$\lambda(T) < \max \left\{ \max_{2 \leq j \leq k-2} \{ \sqrt{\sigma_j - 1} + \sigma_j + \sqrt{\sigma_{j-1} - 1} \}, \sqrt{\sigma_1 - 1} + \sigma_1 + \sqrt{\Delta}, \Delta + \sqrt{\Delta} \right\}. \quad (47)$$

Let G be a graph with edge set E . Two distinct edges in E are said to be *independent* if they are not incident with a common vertex of G . A subset Y of E is called a *matching* of G if the edges in Y are pairwise independent. The *matching number* of G , denoted by $\beta(G)$, is the maximum size of a matching of G .

Guo in [14, Theorem 1] proved the following upper bound (48) for the Laplacian spectral radius of a tree relating to its matching number.

Theorem 3.3. [14] *Let T be a tree with n vertices and matching number $\beta = \beta(T)$. Let κ be the largest root of the equation $x^3 - (n - \beta + 4)x^2 + (3(n - \beta) + 4)x - n = 0$. Then*

$$\lambda(T) \leq \kappa, \quad (48)$$

and equality holds if and only if T is the tree obtained from the star graph $K_{1, n-\beta}$ by adding pendant edges to $\beta - 1$ of the $n - \beta$ pendant vertices of $K_{1, n-\beta}$.

Note that the tree obtained from $K_{1, n-\beta}$ in the equality case of (48) is well-defined, since $\beta(T_1) \leq \lfloor n/2 \rfloor$ for any tree T_1 with n vertices.

Let G be a graph with vertex set V . A subset U of V is called an *independent set* of G if no two vertices of U are adjacent in G . The *independence number* of G , denoted by $\alpha(G)$, is the maximum size of an independent set of G .

In the spirit of Theorem 3.3, Zhang proved in [57, Theorem 2.6, Lemma 2.7] the following upper bound (49) for a tree relating to its independence number.

Theorem 3.4. [57] *Let T be a tree with n vertices and independence number $\alpha = \alpha(T)$. Let ξ be the largest root of the equation $x^3 - (\alpha + 4)x^2 + (3\alpha + 4)x - n = 0$. Then*

$$\lambda(T) \leq \xi, \quad (49)$$

and equality holds if and only if T is the tree obtained from the star graph $K_{1, \alpha}$ by adding pendant edges to $n - \alpha - 1$ of the α pendant vertices of $K_{1, \alpha}$.

The tree obtained from $K_{1,\alpha}$ in the equality case of (49) is well-defined, since $\alpha(T_1) \geq \lceil n/2 \rceil$ for any tree T_1 with n vertices. As an application of Theorem 3.4, Zhang obtained the following bound (50) for a tree in terms of its independence number [57, Corollary 2.8, Theorem 2.9].

Theorem 3.5. [57] *Let T be a tree with n vertices and independence number $\alpha = \alpha(T)$. Then*

$$\lambda(T) \leq \alpha + 1 + \frac{n - \alpha - 1}{(\alpha - 1)^2} < 2 + \alpha. \quad (50)$$

Further, equality holds in the first inequality if and only if T is the star graph $K_{1,n-1}$.

3.2. Non-regular graphs

The following four upper bounds (51) – (54) for the Laplacian spectral radius of non-regular connected graphs are known. The bound (51) was given by Shi in [46, Theorem 3.5].

Theorem 3.6. [46] *Let G be a connected non-regular graph on n vertices with diameter D and maximal vertex degree Δ . Then*

$$\lambda(G) < 2\Delta - \frac{2}{(2D + 1)n}. \quad (51)$$

For a connected graph G , its diameter is always less than the number of vertices. In that case, Theorem 3.6 implies the following [46, Theorem 3.4].

Theorem 3.7. [46] *Let G be a connected non-regular graph with n vertices and maximal vertex degree Δ . Then*

$$\lambda(G) < 2\Delta - 2/n(2n - 1). \quad (52)$$

The bound (53) below, which is an improvement of (51), was given by Liu and Lu in [27, Theorem 2.1].

Theorem 3.8. [27] *Let G be a connected non-regular graph with n vertices, m edges, diameter D and maximal vertex degree Δ . Then*

$$\lambda(G) < 2\Delta - \frac{2n\Delta - 4m}{n(D(2n\Delta - 4m) + 1)}. \quad (53)$$

In [24, Theorem 2.3], Li et al. gave the following bound (54) which further improves the bound (53).

Theorem 3.9. [24] *Let G be a connected non-regular graph with n vertices, diameter D and maximal vertex degree Δ . Then*

$$\lambda(G) < 2\Delta - \frac{1}{nD}. \quad (54)$$

3.3. Triangular graphs

A graph G is called a *triangular graph* if every pair of adjacent vertices of G has at least one common neighbour vertex. The following upper bound (55) for triangular graphs was obtained by Lu et al. in [31, Theorem B].

Theorem 3.10. [31] *Let G be a graph with n vertices, m edges, maximal vertex degree Δ and minimal vertex degree δ . If G is triangular such that each edge of G belongs to at least $t \geq 1$ triangles, then*

$$\lambda(G) \leq \frac{1}{2} \left(2\Delta - t + \sqrt{(2\Delta - t)^2 + 8m - 4\delta(n - 1) - 4\delta^2 + 4(\delta - 1)\Delta} \right), \quad (55)$$

and equality occurs if G is the complete graph K_{t+2} .

Guo et al. proved in [17, Theorem 3.2] the following upper bound (56) which improves (55). By [17, Remark 3.1], (56) is always better than (23) for triangular graphs.

Theorem 3.11. [17] *Let G be a graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . If G is triangular such that each edge of G belongs to at least $t \geq 1$ triangles, then*

$$\lambda(G) \leq \max \left\{ \frac{1}{2} \left(2d_i - t + \sqrt{4d_i m_i - 4t d_i + t^2} \right) : v_i \in V \right\}, \quad (56)$$

and equality occurs if G is the complete graph K_{t+2} .

3.4. Maximal planar graphs

A planar graph G is called a *maximal planar graph* if for every pair of nonadjacent vertices v_i and v_j of G , the graph $G + v_i v_j$ is non-planar. In a maximal planar graph G , each pair of adjacent vertices has at least two common neighbour vertices and so G is a triangular graph in particular.

Taking $t = 2$ in Theorems 3.10 and 3.11, the following two upper bounds (57) and (58) are obtained for maximal planar graphs. The bound (57) appeared in [31, Theorem C], and (58) appeared in [17, Theorem 3.3] which is an improvement of (57).

Theorem 3.12. [31, 17] *Let G be a maximal planar graph with $n \geq 4$ vertices, m edges, vertex degrees d_i , average 2-degrees m_i , maximal vertex degree Δ and minimal vertex degree δ . Then*

$$\lambda(G) \leq \Delta - 1 + \sqrt{(\Delta - 1)^2 + 2m - \delta(n - 1) - \delta^2 + (\delta - 1)\Delta}; \quad (57)$$

and

$$\lambda(G) \leq \max \left\{ d_i - 1 + \sqrt{d_i m_i - 2d_i + 1} : 1 \leq i \leq n \right\}. \quad (58)$$

3.5. Triangle-free graphs

The following two upper bounds (59) and (60) were given by Li et al. in [24, Theorem 3.2, Corollary 3.3] for triangle-free graphs.

Theorem 3.13. [24] *Let G be a triangle-free graph with n vertices, m edges and maximal vertex degree Δ . Then*

$$\lambda(G) \leq \Delta + \sqrt{m}; \quad (59)$$

and

$$\lambda(G) \leq \Delta + \frac{n}{2}. \quad (60)$$

In both the bounds, equality holds if G is the complete bipartite graph $K_{\Delta, \Delta}$.

The bound (60) follows from (59) using Turán's theorem which says that the number of edges in any triangle-free graph is at most $n^2/4$.

3.6. Bipartite graphs

For a bipartite graph $G = (V, E)$ with bipartition $V = V_1 \cup V_2$, let Δ_1 (respectively, Δ_2) denote the maximal vertex degree among the vertices in V_1 (respectively, V_2). Similarly, δ_1 and δ_2 are defined for the minimal vertex degrees in V_1 and V_2 , respectively. The following upper bound was given by Li et al. in [24, Theorem 3.9] for connected graphs (however, connectedness of the graph is not required in the proof).

Theorem 3.14. [24] *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with m edges. If $|V_1| = n_1$ and $|V_2| = n_2$, then*

$$\lambda(G) \leq \max \{ \theta_1, \theta_2 \}, \quad (61)$$

where

$$\begin{aligned} \theta_1 &= \frac{1}{2} \left(\Delta_1 + \delta_2 + \sqrt{(\Delta_1 + \delta_2)^2 + 8(m - n_2\delta_2)} \right); \\ \theta_2 &= \frac{1}{2} \left(\Delta_2 + \delta_1 + \sqrt{(\Delta_2 + \delta_1)^2 + 8(m - n_1\delta_1)} \right). \end{aligned}$$

Further, equality holds if and only if G is semiregular.

4. Lower Bounds

Unlike many upper bounds, only few lower bounds are known for the Laplacian spectral radius of a graph. For any graph G , $\lambda(G) \geq 0$, and equality holds if and only if G has no edge. Recall that we are assuming all our graphs G to be without isolated vertices and hence at least one edge, so that $\lambda(G) > 0$.

The following bound (62) which is the first lower bound for the Laplacian spectral radius of a graph was given by Fiedler in 1973 [11, 3.7(5°)].

Theorem 4.1. [11] *Let G be a graph with n vertices and maximal vertex degree Δ . Then*

$$\lambda(G) \geq \frac{n}{n-1} \Delta. \quad (62)$$

In [12, Corollary 2], Grone and Merris obtained the following lower bound (63) which improves (62). In [60, Theorem 2.3], Zhang and Luo gave an alternate proof of (63) and characterized the equality case for connected graphs.

Theorem 4.2. [12, 60] *Let G be a graph with n vertices and maximal vertex degree Δ . Then*

$$\lambda(G) \geq \Delta + 1. \quad (63)$$

If $\Delta = n - 1$, then equality holds. Conversely, if G is connected and equality holds, then $\Delta = n - 1$.

The following lower bound (64) was given by Zhang and Li in [58, Theorem 2.1] in terms of number of vertices and edges of a graph.

Theorem 4.3. [58] *Let G be a graph with n vertices and m edges. Then*

$$\lambda(G) \geq \frac{1}{n-1} \left(2m + \sqrt{\frac{2m(n(n-1) - 2m)}{n(n-2)}} \right). \quad (64)$$

If G is connected, then equality holds if and only if G is the complete graph K_n .

In [6, Theorem 2.4], Das gave the following lower bound (65) which is an improvement of the bound (63).

Theorem 4.4. [6] *Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then*

$$\lambda(G) \geq \max_{v_i v_j \in E} \left\{ \frac{1}{\sqrt{2}} \left(\sqrt{d_i^2 + 2d_i - 2d_j - 2} + \sqrt{(d_i^2 + 2d_i + 2d_j + 4)^2 + 4r_i r_j} \right) \right\} \quad (65)$$

where $r_i = d_i - c_{ij} - 1$, $r_j = d_j - c_{ij} - 1$ and c_{ij} is the number of common neighbours of v_i and v_j .

The following bound (66) was given by Lu et al. in [29, Lemma 3]. The original result was stated for connected graphs, but their proof works for any graph.

Theorem 4.5. [29] *Let $G = (V, E)$ be a graph with n vertices and $H = (V_1, E_1)$ be an induced subgraph of G with t vertices, where $t < n$. Let $a = \sum_{v \in V_1} d_H(v)/t$ and $b = \sum_{v \in V_1} d_G(v)/t$. Then*

$$\lambda(G) \geq \frac{n(b-a)}{n-t}. \quad (66)$$

If equality holds, then $|K_1(u)|$ is independent of $u \in V_1$ and $|K_2(u)|$ is independent of $u \in V_2$, where $V_2 = V \setminus V_1$, $K_i(u) = \{v \in V_j : uv \in E\}$ for $u \in V_i$, $\{i, j\} = \{1, 2\}$.

The following lower bound (67) was given by Lu et al. in [32, Theorem 3].

Theorem 4.6. [32] Let G be a graph and $P = v_1 v_2 \cdots v_{s+1}$ be a path in G such that the induced subgraph of G on the vertices v_1, v_2, \dots, v_{s+1} is P itself. Then

$$\lambda(G) \geq \frac{1}{s+1} \left(\sum_{i=1}^{s+1} d_i + 2s \right) \quad (67)$$

As an application of Theorem 4.6, the following lower bound (68) was obtained in [32, Corollary 6] for connected graphs.

Theorem 4.7. [32] Let G be a connected graph with n vertices, degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and diameter D . Then

$$\lambda(G) \geq \frac{(D+1)e_{D+1} + 2D}{D+1}, \quad (68)$$

where $e_{D+1} = \frac{1}{D+1} (d_{n-D} + d_{n-D+1} + \cdots + d_n)$. If G is a complete graph, then equality holds in (68).

As consequences of Theorem 4.7, the following lower bounds (69) and (70) were obtained in [32, Corollaries 8, 9]. The bound (70) is better than (69) if the minimal vertex degree is at least two.

Theorem 4.8. [32] Let G be a connected graph with diameter D and minimal vertex degree δ . Then

$$\lambda(G) \geq \frac{4D}{D+1}, \quad (69)$$

and

$$\lambda(G) \geq \frac{(D+1)\delta + 2D}{D+1}. \quad (70)$$

The above lower bounds are given in terms of the number of vertices, vertex degrees, diameter, maximal and minimal vertex degrees of a graph. Some lower bounds for the Laplacian spectral radius of a graph are given involving other graph parameters like independence number, domination number and covering number of the graph.

We have defined the independence number of a graph in Section 3.1. In [57, Theorems 3.1, 3.2], Zhang proved the following lower bounds (71) – (73) involving independence number of a graph.

Theorem 4.9. [57] Let G be a graph with n vertices and independence number $\alpha = \alpha(G)$. Then

$$\lambda(G) \geq \frac{n}{\alpha}, \quad (71)$$

with equality if and only if α is a factor of n and G has α components each of which is the complete graph $K_{\frac{n}{\alpha}}$.

Theorem 4.10. [57] Let G be a graph with n vertices and independence number $\alpha = \alpha(G)$. Then

$$\lambda(G) \geq \frac{n\Sigma_\alpha(G)}{\alpha(n-\alpha)}, \quad (72)$$

where $\Sigma_\alpha(G) = \max \left\{ \sum_{i=1}^{\alpha} d(u_i) : \{u_1, \dots, u_\alpha\} \text{ is an independent set of } G \right\}$.

By our assumption, the minimal vertex degree δ of G is at least one. As an application of (72), it follows using the fact $\Sigma_\alpha(G) \geq \alpha\delta$ that $\lambda(G) \geq \frac{n\delta}{n-\alpha} \geq \frac{n}{n-\alpha}$. In particular,

$$\lambda(G) \geq \frac{n}{\min\{\alpha, n-\alpha\}}. \quad (73)$$

Further, equality holds in (73) if and only if either α is a factor of n and G has α components each of which is the complete graph $K_{\frac{n}{\alpha}}$, or $n-\alpha$ is a factor of n and G has $n-\alpha$ components each of which is the star graph $K_{1, \frac{\alpha}{n-\alpha}}$ [57, Colollary 3.4].

A *dominating set* of a graph G is a subset X of the vertex set V of G such that each vertex of $V \setminus X$ is adjacent to at least one vertex of X . The minimum cardinality of a dominating set is called the *domination number* of G , denoted by $\gamma(G)$. We have $\gamma(G) > 0$ (since V is nonempty).

In [29, Theorem 10], Lu et al. proved the following bound (74) in terms of the number of vertices and the domination number of a connected graph (however, their proof could work for disconnected graphs also).

Theorem 4.11. [29] Let G be a graph with n vertices and domination number $\gamma = \gamma(G)$. Then

$$\lambda(G) \geq n/\gamma. \quad (74)$$

If G is connected, then equality holds if and only if $\Delta(G) = n-1$.

The following lower bound (75) was given by Nikiforov [40, Theorem 3] which slightly improves the bound (74).

Theorem 4.12. [40] Let G be a graph with n vertices and domination number $\gamma = \gamma(G)$. Then

$$\lambda(G) \geq \lceil n/\gamma \rceil. \quad (75)$$

Further, equality holds if and only if $G = G_1 \cup G_2$, where the graphs G_1 and G_2 satisfy the conditions: (i) $|G_1| = \lceil n/\gamma \rceil$ and $\gamma(G_1) = 1$; (ii) $\gamma(G_2) = \gamma - 1$ and $\lambda(G_2) \leq \lceil n/\gamma \rceil$.

A *cover* of a graph G is a subset X of the vertex set of G such that every edge of G is incident with at least one vertex of X . The minimum cardinality of a cover is called the *covering number* of G , denoted by $\tau(G)$. Since all graphs are without isolated vertices by our assumption, it follows that every cover of G is also a dominating set of G . So $\gamma(G) \leq \tau(G)$. Using Theorem 4.11, the following lower bound follows [29, Corollary 11].

Theorem 4.13. [29] Let G be a graph with n vertices and covering number $\tau = \tau(G)$. Then

$$\lambda(G) \geq n/\tau. \quad (76)$$

If G is connected, then equality holds if and only if G is the star graph.

The bound (76) was improved by Shi in [46, Theorem 3.6].

Theorem 4.14. [46] Let G be a graph with n vertices, minimal vertex degree $\delta \geq 1$ and covering number $\tau = \tau(G)$. Then

$$\lambda(G) \geq \delta n/\tau. \quad (77)$$

4.1. Trees

The following lower bound (78) for a tree was given by Das in [8, Theorem 2.4], which is an improvement of (63) in the case of trees.

Theorem 4.15. [8] Let $T = (V, E)$ be a tree with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(T) \geq \max \left\{ \frac{1}{2} \left[(d_i + m_i + 1) + \sqrt{(d_i + m_i + 1)^2 - 4(d_i m_i + 1)} \right] : v_i \in V \right\}. \quad (78)$$

Further, equality holds in (78) if and only if T is isomorphic to the tree $T(d_i, d_j)$ formed by joining the centers of d_i copies of K_{1, d_j-1} to a new vertex w_i .

4.2. Triangle-free graphs

In [60, Theorems 3.1, 3.4], Zhang and Luo proved the following two lower bounds for triangle-free graphs: one is in terms of the number of vertices and edges, and the other is in terms of vertex degrees and average 2-degrees.

Theorem 4.16. [60] Let $G = (V, E)$ be a triangle-free graph with n vertices, m edges, vertex degrees d_i and average 2-degrees m_i . Then the following hold:

$$\lambda(G) \geq \max \left\{ \frac{16m^2}{n^3}, \frac{2m}{n} + \frac{m^{3/4}}{2n\sqrt{2}} \right\}; \quad (79)$$

$$\lambda(G) \geq \max \left\{ \frac{1}{2} \left(d_i + m_i + \sqrt{(d_i - m_i)^2 + 4d_i} : v_i \in V \right) \right\}. \quad (80)$$

Moreover, equality holds in (79) if n is even and G is the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$.

As a consequence of the above two bounds it follows that $\lambda(G) \geq \max \left\{ \frac{4k^2}{n}, k + \sqrt{k} \right\}$ for a k -regular triangle-free graph G on n vertices [60, Corollary 3.5]. By the remark in [60, p.38], (80) is better than (63) for triangle-free graphs.

4.3. Bipartite graphs

We now state the lower bounds known for bipartite graphs. The first result in this direction was obtained by Yu et al. in [54, Theorem 9].

Theorem 4.17. [54] Let $G = (V, E)$ be a bipartite graph with $V = \{v_1, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(G) \geq \sqrt{\left(\sum_{v_i \in V} (d_i^2 + m_i d_i)^2\right) / \left(\sum_{v_i \in V} d_i^2\right)}. \quad (81)$$

If G is connected, then equality holds if and only if G is semiregular.

As an application of Theorem 4.17 together with the Cauchy-Schwarz inequality, the following bound is obtained in [54, Corollary 10], also see [20, Theorem 3.2].

Theorem 4.18. [54] Let G be a bipartite graph with $V = \{v_1, v_2, \dots, v_n\}$ and vertex degrees d_i . Then

$$\lambda(G) \geq 2 \sqrt{\frac{1}{n} \sum_{v_i \in V} d_i^2}. \quad (82)$$

If G is connected, then equality holds if and only if G is regular.

From the above theorem, it follows that $\lambda(G) \geq \frac{4m}{n} \geq 2\delta$ for a bipartite graph G , where G has n vertices, m edges and minimal vertex degree δ [54, Corollary 11]. The following lower bound was given by Hong and Zhang in [20, Theorem 3.3].

Theorem 4.19. [20] Let $G = (V, E)$ be a bipartite graph with $V = \{v_1, \dots, v_n\}$, m edges and vertex degrees d_i . Then

$$\lambda(G) \geq 2 + \sqrt{\frac{1}{m} \sum_{v_i v_j \in E} (d_i + d_j - 2)^2}. \quad (83)$$

If G is connected, then equality holds if and only if G is a semiregular graph or a path with four vertices.

In [46, Theorems 3.1, 3.3, Corollary 3.1], Shi gave the following three lower bounds for the Laplacian spectral radius of bipartite graphs.

Theorem 4.20. [46] Let $G = (V, E)$ be a bipartite graph with $V = \{v_1, v_2, \dots, v_n\}$, m edges, vertex degrees d_i , average 2-degrees m_i , maximal vertex degree Δ and minimal vertex degree δ . Then the following hold:

$$\lambda(G) \geq \min \left\{ \sqrt{2d_i(d_i + m_i)} : v_i \in V \right\}; \quad (84)$$

$$\lambda(G) \geq \sqrt{2\delta^2 + 4m - 2\Delta(n-1) + 2\delta(\Delta-1)}; \quad (85)$$

$$\lambda(G) \geq \left[\sum_{v_i \in V} \left(d_i^{3/2} + \sum_{v_i v_j \in E} \sqrt{d_j} \right)^2 / (2m) \right]^{1/2}. \quad (86)$$

If G is connected, then equality holds in each of these three bounds if and only if G is regular.

Though connectedness is assumed in [46, Theorems 3.3] for the proof of the bound (86), a careful application of Lemma 1.2 shows that the given proof would work for disconnected graphs also. The equality case of (84) in [46, Theorem 3.1] was characterized as the following: If the bipartite graph G is connected, then equality holds in (84) if and only if the value $d_i^2 + d_i m_i$ is independent of the vertex $v_i \in V$, but this is equivalent to that G is regular (see Remark 2.1).

The following bound was given by Guo in [16, Theorem 4.1]. The same bound was also obtained in [24, Theorem 3.6], where equality case was characterized. Their proofs work for any bipartite graph.

Theorem 4.21. [16, 24] Let $G = (V, E)$ be a bipartite graph with n vertices and m edges. Suppose that V has bipartition $V = V_1 \cup V_2$ with $|V_1| = n_1$ and $|V_2| = n_2$. Then

$$\lambda(G) \geq \frac{mn}{n_1 n_2}. \quad (87)$$

Further, equality holds if and only if G is semiregular.

The following bound was given in [24, Theorem 3.5] for non-regular bipartite graphs in terms of the number of vertices and edges.

Theorem 4.22. [24] Let G be a non-regular bipartite graph with n vertices and m edges. Then

$$\lambda(G) \geq \frac{4m}{n} + \frac{1}{m+n}. \quad (88)$$

The next bound was given by Tian et al. in [50, Theorem 1], as a corollary of which the bound (81) was obtained in [50, Corollary 2].

Theorem 4.23. [50] Let $G = (V, E)$ be a bipartite graph with $V = \{v_1, v_2, \dots, v_n\}$, vertex degrees d_i and average 2-degrees m_i . Then

$$\lambda(G) \geq \sqrt{\sum_{i=1}^n \left(d_i(d_i^2 + m_i d_i) + \sum_{v_i v_j \in E} (d_j^2 + m_j d_j) \right)^2 / \left(\sum_{i=1}^n (d_i^2 + m_i d_i)^2 \right)}. \quad (89)$$

If G is connected, then equality holds if and only if there exists a positive constant t such that

$$\frac{d_i(d_i^2 + m_i d_i) + \sum_{v_i v_j \in E} (d_j^2 + m_j d_j)}{d_i^2 + m_i d_i} = t$$

for all $i \in \{1, 2, \dots, n\}$. In fact $t = \lambda(G)$.

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