



# Perfect 3-colorings of the cubic graphs of order 10

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## Abstract

Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect  $m$ -coloring of a graph  $G$  with  $m$  colors is a partition of the vertex set of  $G$  into  $m$  parts  $A_1, A_2, \dots, A_m$  such that, for all  $i, j \in \{1, \dots, m\}$ , every vertex of  $A_i$  is adjacent to the same number of vertices, namely,  $a_{ij}$  vertices, of  $A_j$ . The matrix  $A = (a_{ij})_{i,j \in \{1, \dots, m\}}$  is called the parameter matrix. We study the perfect 3-colorings (also known as the equitable partitions into three parts) of the cubic graphs of order 10. In particular, we classify all the realizable parameter matrices of perfect 3-colorings for the cubic graphs of order 10.

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## 1. Introduction

The concept of a perfect  $m$ -coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see[10]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done

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on enumerating the parameter matrices of some Johnson graphs, including  $J(4, 2)$ ,  $J(5, 2)$ ,  $J(6, 2)$ ,  $J(6, 3)$ ,  $J(7, 3)$ ,  $J(8, 3)$ ,  $J(8, 4)$ , and  $J(v, 3)$  ( $v$  odd) (see [1, 3, 4, 9]).

Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of  $n$ -dimensional hypercube  $Q_n$  for  $n < 24$ . He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the  $n$ -dimensional cube with a given parameter matrix (see [6, 7, 8]). In this paper all graphs are assumed simple, connected and undirected. First we give some basic definitions and concepts. Let  $G = (V, E)$  be a graph. Two vertices  $u, v \in V(G)$  are adjacent if there exists an edge  $e = \{u, v\} \in E(G)$  to which they are both incident. The adjacent will be shown  $u \leftrightarrow v$ .

A cubic graph is a 3-regular graph. In [5], it is shown that the number of connected cubic graphs with 10 vertices is 19. Each graph is described by a drawing as shown in Figure 1.

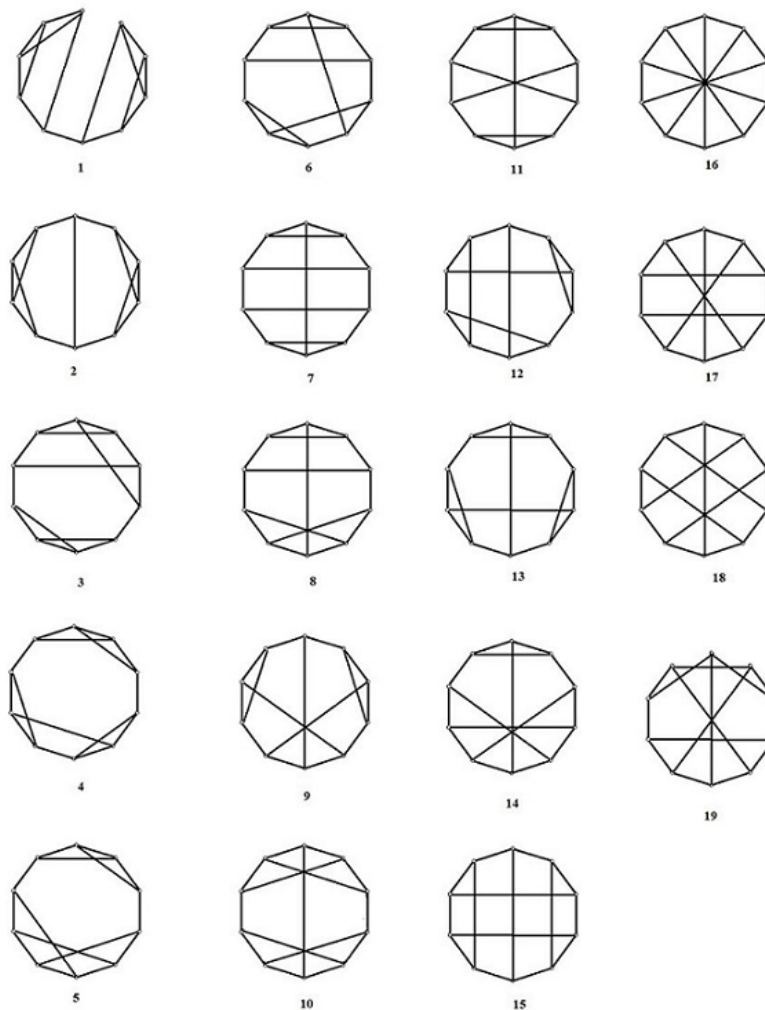


Figure 1. Connected cubic graphs of order 10.

**Definition 1.1.** For a graph  $G$  and a positive integer  $m$ , a mapping  $T : V(G) \rightarrow \{1, \dots, m\}$  is called a perfect  $m$ -coloring with matrix  $A = (a_{ij})_{i,j \in \{1, \dots, m\}}$ , if it is surjective, and for all  $i, j$ , for every vertex of color  $i$ , the number of its neighbors of color  $j$  is equal to  $a_{ij}$ . The matrix  $A$  is called the *parameter matrix* of a perfect coloring. In the case  $m = 3$ , we call the first color *white*, the second color *black*, and the third color *red*. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

*Remark 1.1.* In this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e. we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix}, \begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

obtained by switching the colors with the original coloring.

## 2. Preliminaries and Analysis

In this section, we present some results concerning necessary conditions for the existence of perfect 3-colorings of a cubic connected graph of order 10 with a given parameter matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix  $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  is:

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have  $b = c = 0, d = f = 0$ , or  $g = h = 0$ , since the graph is connected. In addition,  $b = 0, c = 0, f = 0$  if  $d = 0, g = 0, h = 0$ , respectively.

The number  $\theta$  is called an eigenvalue of a graph  $G$ , if  $\theta$  is an eigenvalue of the adjacency matrix of this graph. The number  $\theta$  is called an eigenvalue of a perfect coloring  $T$  into three colors with the matrix  $A$ , if  $\theta$  is an eigenvalue of  $A$ . The following lemma demonstrates the connection between the introduced notions.

**Lemma 2.1.** [10] *If  $T$  is a perfect coloring of a graph  $G$  in  $m$  colors, then any eigenvalue of  $T$  is an eigenvalue of  $G$ .*

Now, without lost of generality, we can assume that  $|W| \leq |B| \leq |R|$ . The following proposition gives us the size of each class of color.

**Proposition 2.1.** Let  $T$  be a perfect 3-coloring of a graph  $G$  with the matrix  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ .

1. If  $b, c, f \neq 0$ , then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

2. If  $b = 0$ , then

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

3. If  $c = 0$ , then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

4. If  $f = 0$ , then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

*Proof. (1):* Consider the 3-partite graph obtained by removing the edges  $uv$  such that  $u$  and  $v$  are the same color. By counting the number of edges between parts, we can easily obtain  $|W|b = |B|d$ ,  $|W|c = |R|g$ , and  $|B|f = |R|h$ . Now, we can conclude the desired result from  $|W| + |B| + |R| = |V(G)|$ .

The proof of (2), (3), (4) is similar to (1). □

In the next lemma, under the condition  $|W| = 1$ , we enumerate all matrices that can be a parameter matrix for a cubic connected graph.

**Lemma 2.2.** Let  $G$  be a cubic connected graph of order 10. If  $T$  is a perfect 3-coloring with the matrix  $A$ , and  $|W| = 1$ , then  $A$  should be the following matrix:

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

*Proof.* Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  be a parameter matrix with  $|W| = 1$ . Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e.  $a = 0$ . Therefore, we have two cases below.

- (1) The adjacent vertices of the white vertex are the same color. If they are black, then  $b = 3$  and  $c = 0$ . From  $c = 0$ , we get  $g = 0$ . Also, since the graph is connected, we have  $f, h \neq 0$ . Hence we obtain the following matrices:

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

If the adjacent vertices of the white vertex are red, then  $c = 3, b = 0$ . From  $b = 0$ , we get  $d = 0$ . Also, since the graph is connected, we have  $f, h \neq 0$ . Hence we obtain the following matrices:

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Finally, by using Remark 1.1 and the fact that  $|W| \leq |B| \leq |R|$ , it is obvious that there are only six matrices in (1), as shown  $A_1, A_2, A_3, A_4, A_5, A_6$ .

$$A_1 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

- (2) The adjacent vertices of the white vertex are different colors. It immediately gives that  $b, c \neq 0$ . Also, it can be seen that  $d = g = 1$ . An easy computation as in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown  $A_7, A_8, A_9, A_{10}, A_{11}$ .

$$A_7 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, A_9 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, A_{10} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

By using Proposition 2.1, it is obvious that just the matrix  $A := A_2$  can be a parameter.

□

**Lemma 2.3.** *Let  $G$  be a cubic connected graph of order 10. If  $T$  is a perfect 3-coloring with the matrix  $A$ , and  $|W| = |B| = 2$ ,  $|R| = 6$ , then  $A$  should be the following matrix*

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

*Proof.* First, suppose that  $b, c \neq 0$ . As  $|W| = 2$ , by Proposition 2.1, it follows that  $\frac{b}{d} + \frac{c}{g} = 4$ . Therefore  $b = c = 2, d = g = 1$  and we get a contradiction with  $b + c \leq 3$ .

Second, suppose that  $b = 0$  and then  $d = 0$ . As  $|R| = 4$ , by Proposition 2.1, we have  $\frac{g}{c} + \frac{h}{f} = \frac{2}{3}$ .

Therefore  $c = f = 3, g = h = 1$ , and consequently  $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ .

Finally, suppose that  $c = 0$  and then  $g = 0$ . As  $|B| = 2$ , by Proposition 2.1, it follows that  $\frac{d}{b} + \frac{f}{h} = 4$ . Therefore  $b = f = 2, d = h = 1$ , or  $b = 3, d = f = h = 1$  or  $b = 3, d = 1,$

$f = h = 2$ . Hence  $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ , or  $A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ , or  $A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ .

By using the Proposition 2.1, it can be seen that only the matrix  $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}$  can be a parameter.  $\square$

**Lemma 2.4.** *Let  $G$  be a cubic connected graph of order 10. Then  $G$  has no perfect 3-coloring  $T$  with the matrix that  $|W| = 2, |B| = 3, |R| = 5$ .*

*Proof.* If  $T$  is a perfect 3-coloring with the similar proving Lemma2.3,  $A$  should be one of the following matrices:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

By using the Proposition 2.1, it can be seen that no matrix can be a parameter.  $\square$

**Lemma 2.5.** *Let  $G$  be a cubic connected graph of order 10. If  $T$  is a perfect 3-coloring with the matrix  $A$ , and also if  $|W| = 2, |B| = 4, |R| = 4$ , then  $A$  should be one of the following matrices:*

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

*Proof.* If  $T$  is a perfect 3-coloring with the similar proving Lemma2.3, then  $A$  should be one of the following matrices:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

By using the Proposition 2.1, it can be seen that the following matrices should be parameter:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

□

**Lemma 2.6.** *Let  $G$  be a cubic connected graph of order 10. Then  $G$  has no perfect 3-coloring  $T$  with the matrix that  $|W| = 3, |B| = 3, |R| = 4$ .*

*Proof.* If  $T$  is a perfect 3-coloring with the similar proving Lemma2.3, then  $A$  should be one of the following matrices:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

By using Proposition 2.1, it can be seen that no matrix can be a parameter. □

By using Lemmas 2.2, 2.3 and 2.5, it can be seen that only the following matrices can be parameter ones.

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

By Remark 1.1, it is clear that the matrix  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  is the same as the matrix  $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$  and the matrix  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$  is the same as the matrix  $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$  up to renaming the colors. Therefore, if  $T$  is a perfect 3-coloring with the matrix  $A$ , then  $A$  should be one of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

The next theorem can be useful to find the eigenvalues of a parameter matrix.

**Theorem 2.1.** Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$  be a parameter matrix of a  $k$ -regular graph. Then the eigenvalues of  $A$  are

$$\lambda_{1,2} = \frac{\text{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\text{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}, \quad \lambda_3 = k.$$

*Proof.* By using the condition  $a + b + c = d + e + f = g + h + i = k$ , it is clear that one of the eigenvalues is  $k$ . Therefore  $\det(A) = k\lambda_1\lambda_2$ . From  $\lambda_2 = \text{tr}(A) - \lambda_1 - k$ , we get

$$\det(A) = k\lambda_1(\text{tr}(A) - \lambda_1 - k) = -k\lambda_1^2 + k(\text{tr}(A) - k)\lambda_1.$$

By solving the equation  $\lambda^2 + (k - \text{tr}(A))\lambda + \frac{\det(A)}{k} = 0$ , we obtain

$$\lambda_{1,2} = \frac{\text{tr}(A) - k}{2} \pm \sqrt{\left(\frac{\text{tr}(A) - k}{2}\right)^2 - \frac{\det(A)}{k}}.$$

□

### 3. Perfect 3-colorings of the cubic connected graphs of order 10

In this section, we enumerate the parameter matrices of all perfect 3-colorings of the cubic connected graphs of order 10.

**Theorem 3.1.** The parameter matrices of cubic graphs of order 10 are listed in the following table.



graphs	matrix $A_1$	matrix $A_2$	matrix $A_3$	matrix $A_4$
1	√	×	√	×
2	√	×	√	√
3	×	×	×	×
4	√	×	×	×
5	×	×	×	×
6	√	×	√	×
7	×	×	×	×
8	×	×	×	×
9	×	×	√	√
10	√	×	√	×
11	×	×	×	×
12	×	×	×	×
13	×	×	×	√
14	×	×	×	√
15	×	×	×	×
16	×	×	×	×
17	×	×	×	×
18	×	×	√	√
19	×	×	√	√

Table 1

*Proof.* As it has been shown in Section 3, only matrices  $A_1, A_2, A_3$  and  $A_4$  can be parameter matrices. With consideration of cubic graphs eigenvalues and using Theorem 2.1, it can be seen that the connected cubic graphs with 10 vertices can have perfect 3-coloring with matrices  $A_1, A_2, A_3$  and  $A_4$  which is represented by Table 2.

graphs	matrix $A_1$	matrix $A_2$	matrix $A_3$	matrix $A_4$
1	√	√	√	√
2	√	√	√	√
4	√	√	×	×
5	√	√	√	√
6	√	√	√	√
9	√	√	√	√
10	√	√	√	√
13	×	×	√	√
14	×	×	√	√
18	√	√	√	√
19	×	×	√	√

Table 2

The vertices of cubic graphs are labeled clockwise with  $a_1, a_2, \dots, a_{10}$ , respectively. The graph 1 has perfect 3-colorings with the matrices  $A_1$  and  $A_3$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_1) &= T_1(a_{10}) = 1, T_1(a_4) = T_1(a_7) = 2, \\ T_1(a_2) &= T_1(a_3) = T_1(a_5) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3. \\ T_2(a_5) &= T_2(a_6) = 1, T_2(a_2) = T_2(a_3) = T_2(a_8) = T_2(a_9) = 2, \\ T_2(a_1) &= T_2(a_4) = T_2(a_7) = T_2(a_3) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_1$  and  $A_3$ , respectively.

The graph 2 has perfect 3-colorings with the matrices  $A_1$ ,  $A_3$  and  $A_4$ . Consider three mappings  $T_1$ ,  $T_2$  and  $T_3$  as follows:

$$\begin{aligned} T_1(a_2) &= T_1(a_7) = 1, T_1(a_5) = T_1(a_{10}) = 2, \\ T_1(a_1) &= T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3. \\ T_2(a_1) &= T_2(a_6) = 1, T_2(a_3) = T_2(a_4) = T_2(a_8) = T_2(a_9) = 2, \\ T_2(a_2) &= T_2(a_5) = T_2(a_7) = T_2(a_{10}) = 3. \\ T_3(a_1) &= 1, T_3(a_2) = T_3(a_6) = T_3(a_{10}) = 2, \\ T_3(a_3) &= T_3(a_4) = T_3(a_5) = T_3(a_7) = T_3(a_8) = T_3(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$ ,  $T_2$  and  $T_3$  are perfect 3-coloring with the matrices  $A_1$ ,  $A_3$  and  $A_4$ , respectively.

The graph 4 has perfect 3-colorings with the matrix  $A_1$ . Consider the mapping  $T_1$  as follows:

$$\begin{aligned} T_1(a_5) &= T_1(a_{10}) = 1, T_1(a_2) = T_1(a_7) = 2, \\ T_1(a_1) &= T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  is a perfect 3-coloring with the matrix  $A_1$ .

The graph 6 has perfect 3-colorings with the matrices  $A_1$  and  $A_3$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_5) &= T_1(a_9) = 1, T_1(a_7) = T_1(a_2) = 2, \\ T_1(a_1) &= T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_{10}) = 3. \\ T_2(a_3) &= T_2(a_4) = 1, T_2(a_1) = T_2(a_6) = 2 = T_2(a_8) = T_2(a_{10}) = 2, \\ T_2(a_2) &= T_2(a_5) = T_2(a_7) = T_2(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_1$  and  $A_3$ , respectively.

The graph 9 has perfect 3-colorings with the matrices  $A_3$  and  $A_4$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_1) &= T_1(a_6) = 1, T_1(a_3) = T_1(a_4) = T_1(a_8) = T_1(a_9) = 2, \\ T_1(a_2) &= T_1(a_5) = T_1(a_7) = T_1(a_{10}) = 3. \\ T_2(a_1) &= 1, T_2(a_2) = T_2(a_6) = 2 = T_2(a_{10}) = 2, \\ T_2(a_3) &= T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_3$  and  $A_4$ , respectively.

The graph 10 has perfect 3-colorings with the matrices  $A_1$  and  $A_3$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_2) = T_1(a_5) = 1, T_1(a_7) = T_1(a_{10}) = 2, \\ T_1(a_1) = T_1(a_3) = T_1(a_4) = T_1(a_6) = T_1(a_8) = T_1(a_9) = 3. \\ T_2(a_1) = T_2(a_6) = 1, T_2(a_3) = T_2(a_4) = 2 = T_2(a_8) = T_2(a_9) = 2, \\ T_2(a_2) = T_2(a_5) = T_2(a_7) = T_2(a_{10}) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_1$  and  $A_3$ , respectively.

The graph 13 has perfect 3-colorings with the matrix  $A_4$ . Consider a mapping  $T_1$  as follows:

$$\begin{aligned} T_1(a_6) = 1, T_1(a_1) = T_1(a_5) = T_1(a_7) = 2, \\ T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = T_1(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  is a perfect 3-coloring with the matrix  $A_4$ .

The graph 14 has perfect 3-colorings with the matrix  $A_4$ . Consider a mapping  $T_1$  as follows:

$$\begin{aligned} T_1(a_6) = 1, T_1(a_1) = T_1(a_5) = T_1(a_7) = 2, \\ T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_7) = T_1(a_8) = T_1(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  is a perfect 3-coloring with the matrix  $A_4$ .

The graph 18 has perfect 3-colorings with the matrices  $A_3$  and  $A_4$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_1) = T_1(a_6) = 1, T_1(a_3) = T_1(a_4) = T_1(a_8) = T_1(a_9) = 2, \\ T_1(a_2) = T_1(a_5) = T_1(a_7) = T_1(a_{10}) = 3. \\ T_2(a_1) = 1, T_2(a_2) = T_2(a_6) = 2 = T_2(a_{10}) = 2, \\ T_2(a_3) = T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_9) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_3$  and  $A_4$ , respectively.

The graph 19 has perfect 3-colorings with the matrices  $A_3$  and  $A_4$ . Consider two mappings  $T_1$  and  $T_2$  as follows:

$$\begin{aligned} T_1(a_2) = T_1(a_9) = 1, T_1(a_1) = T_1(a_4) = T_1(a_6) = T_1(a_8) = 2, \\ T_1(a_3) = T_1(a_5) = T_1(a_7) = T_1(a_9) = 3. \\ T_2(a_1) = 1, T_2(a_3) = T_2(a_6) = 2 = T_2(a_9) = 2, \\ T_2(a_2) = T_2(a_4) = T_2(a_5) = T_2(a_7) = T_2(a_8) = T_2(a_{10}) = 3. \end{aligned}$$

It is clear that  $T_1$  and  $T_2$  are perfect 3-coloring with the matrices  $A_3$  and  $A_4$ , respectively. There are no perfect 3-colorings with the matrices  $A_2$  and  $A_4$  for graph 1.

Contrary to our claim, suppose that  $T$  is a perfect 3-coloring with the matrix  $A_2$  for graph 1. According to the matrix  $A_2$ , each vertex with white color has a neighbor with white color, so the two vertices with white color are adjacent. In the case that  $a_1 \leftrightarrow a_2, a_1 \leftrightarrow a_3, a_2 \leftrightarrow a_4, a_3 \leftrightarrow a_4$  by symmetry  $a_7 \leftrightarrow a_8, a_7 \leftrightarrow a_9, a_8 \leftrightarrow a_{10}$  and  $a_9 \leftrightarrow a_{10}$ , they have less than four adjacent vertices. These vertices are red color, which is a contradiction. So  $a_5 \leftrightarrow a_6, a_4 \leftrightarrow a_5$  and its symmetric  $a_6 \leftrightarrow a_7$  will be remain that are white color. In the case that  $a_4 \leftrightarrow a_5$ , the neighbors of  $a_4$  and  $a_5$  are red color and vertex  $a_1$  that is their neighbor's is also red color has two neighbors with red color which it is not possible. If  $a_5$  and  $a_6$  are white color, adjacent vertices are red color and other vertices are black color, so each black color is adjacent to another black color vertex, which is a contradiction. So we conclude the graph 1 has no perfect 3-coloring with matrix  $A_2$ .

Contrary to our claim, suppose that  $T$  is a perfect 3-coloring with the matrix  $A_4$  for graph 1. According to the matrix  $A_4$ , each vertex with white color has three adjacent with black color. If  $a_1$  is white color, then  $a_2, a_3, a_5$  are black color, which is a contradiction with the second row of matrix  $A_4$ . If  $a_2$  is white color, then according to the matrix  $A_4$ , the vertices  $a_1, a_3, a_4$  are black color, which is a contradiction with the second row of matrix  $A_4$ . If  $a_3$  is white color, then according to the matrix  $A_4$ , the vertices  $a_1, a_2, a_4$  are black color, which is a contradiction with the second row of matrix  $A_4$ . If  $a_4$  is white color, then according to the matrix  $A_4$ , the vertices  $a_2, a_3, a_5$  are black color, which is a contradiction with the second row of matrix  $A_4$ . If  $a_5$  is white color, then  $a_3$  is a vertex that is black color and has three red color neighbors, which is a counteraction with the second row of matrix  $A_4$ . According to the symmetric, the vertices  $a_6, a_7, a_8, a_9, a_{10}$  can not be white color. Therefore the graph 1 has no perfect 3-coloring with matrix  $A_4$ .

As it is stated in the before paragraphs, the graph 1 has no perfect 3-coloring with matrices  $A_2$  and  $A_4$ .

About other graphs in Figure 1, similarly, we can get the same result as in Table 1. □

## References

- [1] M. Alaeiyan and A. Abedi, Perfect 2-colorings of Johnson graphs  $J(4, 3)$ ,  $J(4, 3)$ ,  $J(6, 3)$  and Petersen graph, *Ars Combinatoria*, (to appear).
- [2] M. Alaeiyan and H. Karami, Perfect 2-colorings of the generalized Petersen graph, *Proceedings Mathematical Sciences* **126** (2016), 1–6.
- [3] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect 2-colorings of Johnson graphs  $J(6, 3)$  and  $J(7, 3)$ , *Lecture Notes in Computer Science* **5228** (2008), 11–19.
- [4] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect colorings of the Johnson graphs  $J(8, 3)$  and  $J(8, 4)$  with two colors, *Journal of Applied and Industrial Mathematics* **5** (2011), 19-30.
- [5] F.C. Bussemaker, S. Cobeljic, D.M. Cvetkovic and J.J. Seidel, *Computer investigation of cubic graphs*, Technische Hogeschool Eindhoven Nederland Onderafdeling Der Wiskunde, January 1976.

- [6] D.G. Fon-Der-Flaass, A bound on correlation immunity, *Siberian Electronic Mathematical Reports Journal* **4** (2007), 133–135.
- [7] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube, *Siberian Mathematical Journal* **4** (2007), 923–930.
- [8] D.G. Fon-der-Flaass, Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity, *Siberian Mathematical Journal* **4** (2007), 292–295.
- [9] A.L. Gavrilyuk and S.V. Goryainov, On perfect 2-colorings of Johnson graphs  $J(v, 3)$ , *Journal of Combinatorial Designs* **21** (2013), 232–252.
- [10] C. Godsil, Compact graphs and equitable partitions, *Linear Algebra and Its Application* **255** (1997), 259–266.
- [11] C. Godsil and R. Gordon, *Algebraic graph theory*, Springer Science+Business Media, LLC (2004).