A Variational Approach of Creeping Solitons with Hartman-Grobman Theorem in Complex Ginzburg-Landau Equation

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Abstract— The behavior of quintic nonlinear dispersion coefficient of creeping soliton in a spatial domain with hyperbolicity analysis of Hartman-Grobman Theorem by using variational approach is studied. Complex Ginzburg-Landau equation (CGLE) is used in the analysis as we relate the creeping soliton with Hartman-Grobman Theorem. We evaluated our work based on perturbed Jacobian matrix from system of three supercritical ordinary differential Euler-Lagrange equations, in which the eigenvalues of the stability matrix touch the imaginary axis. As a consequence in unfolding the bifurcation of creeping solitons, the equilibrium structure ultimately chaotic at the variation of the coefficient μ away from the critical value, μ_c. This leads to hyperbolicity loss of Hartman-Grobman Theorem in the dissipative system driven out the oscillatory instability of μ exceeded the criticality parameter corresponding to the Hopf bifurcations as the system is highly complex. This overall approach restrict to numerical investigation of the space time hyperbolic variation of CGLE.

Keywords— Dissipative solitons, complex Ginzburg-Landau equation.

1   INTRODUCTION

The Complex Ginzburg-Landau equation is a generic equation that models a variety of phenomena in weakly nonlinear behavior of dissipative systems. We are dealing with infinite dimensional dynamical system governed by the CGLE throughout this paper. The variety of these systems is so large that "The World of the Ginzburg-Landau equation" is about broadly list of applications in the equation, written by Aranson and Kramer [1]. The CGLE mostly has a wide range of applications in various branches of sciences (physic, biology and chemistry) [2]. Other than that, this complex equation is a model of generation in superconductivity, super fluidity, Bose-Einstein condensation, and strings in field theory. Depending on the system parameters, the CGLE has different types of solutions, including solitons, fronts, pulsating, snaking [3], and creeping, erupting, and chaotic solitons [2]. There is a practical difficulty of finding a parallel between different regions in the parameter space and various types of localized waves since the CGLE is characterized by several parameters. However, the solution to this problem usually needs huge numerical simulations with different sets of parameters and initial conditions.

A creeping soliton is a special type of pulsating localized solution that changes its shape periodically and shifts a finite distance in the transverse direction after each period of oscillations [4]. Although the motion occurs as a step-by-step translation in one direction, the value of the shift is constant for each period so that the soliton has a finite average velocity. For instance, creeping solitons have long flat-top profiles that consist of two fronts (move asymmetrically in time) at the sides of the soliton which the two fronts are creating creeping movements of the whole 'worm-like' formation.

Creeping solitons were first observed in numerical simulations by Soto-Crespo et. al [5]. Their existence has been confirmed in other publications for various dissipative systems [6][7][8].

Creeping solitons exist in a range of the equation parameters. Only isolated examples had been found previously [8].The results of simulations are summarized in [2]. Here, we would like to particularly site Mancas's work of the variational formulation in CGLE [9]. We attempt to extend our attention on his method and use to relate with Hartman-Grobman Theorem. Since the variational formalism not so well explored by others, this study only approach in a novel way.

In this paper, we address the issues of study of behavior of 5th order nonlinear dispersion of creeping soliton in the CGLE with hyperbolicity analysis of Hartman-Grobman Theorem by using variational approach. To analyze the hyperbolicity of creeping solitons, we relate it to the Hartman-Grobman Theorem. We analyzed our problem based on perturbed variational eigenvalues approach in the reduced supercritical ordinary differential equations (ODEs) in the Euler-Lagrange system, which the eigenvalues of the Jacobian matrix touch the imaginary axis. Taking μ as a control bifurcation parameter, we can restrict ourselves to analyze the numerical investigation of the space-time hyperbolic variation of the CGLE.

2 MATHEMATICAL MODEL

In our notation, the complex GLE can be written as

$$\partial_t A = \delta A + (\beta + i\frac{D}{2})\partial_x^2 A - (\varepsilon - i\gamma)|A|^2 A - (\mu - i\nu)|A|^4 A \qquad (1)$$

where t and x are distance traveled variable and retarded time, respectively. Note that the coefficients can be set to unity by appropriate scaling of time, space and $A(x,t)$. D is group velocity dispersion coefficient, with $D = \pm 1$, depending on whether the group velocity is anomalous or normal, respectively, δ is the linear gain-loss coefficient, β is spectral filtering or linear parabolic gain $(\beta > 0)$, ε represents the nonlinear gain, μ is the saturation of the nonlinear dispersion, ν is the coefficient of quintic nonlinearity, and γ is a higher order correction of the cubic nonlinearity. In this paper, we shall assume $D = 1$ and $\gamma = 1$ [5].

3 HYPERBOLIC ANALYSIS

If one of the parameters changes, it causes pulsating solitons to exhibit more complicated behaviors. These pulsations can be transformed by period-doubling bifurcations of creeping solitons as the parameter changes further. This is due to the bifurcation at certain boundaries in the parameter space and also through a sequence of period-doubling bifurcation.

Since Mancas [9] has succeeded on the work of variational method in pulsating and snaking solitons, hence we decide to study another soliton class by analyzing its hyperbolicity. This investigation can be related to Hartman-Grobman Hyperbolic Theorem.

The fixed point of Euler-Lagrange equations in system are as follow:

$$\dot{A_1}(t) = f_1[A_1(t), \varphi(t), \alpha(t)]$$

$$\dot{\varphi}(t) = f_2[A_1(t), \varphi(t), \alpha(t)] \qquad (2)$$

$$\dot{\alpha}(t) = f_3[A_1(t), \varphi(t), \alpha(t)]$$

For the systems of differential equations given by (2), by using progressively slower spatial scales, the limit cycle is determined by expanding the amplitude $A_1(t)$, the inverse width $\phi(t)$ and the phases of solitons $\alpha(t)$. All are allowed to vary arbitrarily in time, and the chirp terms of trial functions are omitted for simplicity [3].

Mancas [9] was introduced θ as the usual multiple scales expansion parameter and set $\theta = 1$ at the end in the usual way. We choose the parameter μ as the control or distinguished bifurcation parameter. The

expansion of fixed point (2) takes the form

$$A_1(t) = A_{11}(Z_0, Z_1, Z_2) + \theta A_{12}(Z_0, Z_1, Z_2) + \theta^2 A_{13}(Z_0, Z_1, Z_2)..., \tag{3}$$

$$\varphi(t) = \varphi_1(Z_0, Z_1, Z_2) + \theta \varphi_2(Z_0, Z_1, Z_2) + \theta^2 \varphi_3(Z_0, Z_1, Z_2)..., \tag{4}$$

$$\alpha(t) = \alpha_1(Z_0, Z_1, Z_2) + \theta \alpha_2(Z_0, Z_1, Z_2) + \theta^2 \alpha_3(Z_0, Z_1, Z_2)... \tag{5}$$

The delay parameter μ is ordered as

$$\mu = \mu_c + \theta \mu_1 \tag{6}$$

where μ_c is the critical parameter value, such that the necessary conditions for characteristics polynomial of Jacobian matrix for fixed point (2) to have $\text{Re}(\lambda) < 0$ is not satisfied, (i.e. μ_c is a solution of one of the conditions) [3]. Using (3)-(5) in (2) and equating powers of θ yields equations at $O(\theta^i)$ of the form:

$$\overrightarrow{S_{i,j}} = \frac{d}{dZ_0}\overrightarrow{x_i} + \begin{pmatrix} f_{1v} & f_{2v} & f_{3v} \\ f_{1w} & f_{2w} & f_{3w} \\ f_{1z} & f_{2z} & f_{3z} \end{pmatrix}\overrightarrow{x_i} \tag{7}$$

where $i = 1,2,3$ represents the order, $j = 1,2,3$ represents the equations, $\overline{S_{i,j}}$ is the source or inhomogeneous terms for the jth equation at $O(\theta^i)$, and $\overline{x_i}$ is the solution according the order which we assume it depend on parameter μ and x hyperbolic equilibrium of μ,

Here, $\begin{pmatrix} f_{1v} & f_{2v} & f_{3v} \\ f_{1w} & f_{2w} & f_{3w} \\ f_{1z} & f_{2z} & f_{3z} \end{pmatrix} = J\left[\dfrac{\partial f_1, \partial f_2, \partial f_3}{\partial A_1, \partial \varphi, \partial \alpha}\right] \tag{8}$

where J is the Jacobian matrix of (2).

The second order sources in standard way, take the form

$$\overrightarrow{S_{2,j}} = \begin{bmatrix} S_{2,1}^0 \\ S_{2,2}^0 \\ S_{2,3}^0 \end{bmatrix} + \begin{bmatrix} S_{2,1}^1 \\ S_{2,2}^1 \\ S_{2,3}^1 \end{bmatrix}e^{-i\omega_0 z_0} + \begin{bmatrix} S_{2,1}^2 \\ S_{2,2}^2 \\ S_{2,3}^2 \end{bmatrix}e^{2i\omega_0 z_0} + c.c, \tag{9}$$

and the third order sources

$$\overrightarrow{S_{3,j}} = \begin{bmatrix} S_{3,1}^0 \\ S_{3,2}^0 \\ S_{3,3}^0 \end{bmatrix} + \begin{bmatrix} S_{3,1}^1 \\ S_{3,2}^1 \\ S_{3,3}^1 \end{bmatrix}e^{-i\omega_0 z_0} + \begin{bmatrix} S_{3,1}^2 \\ S_{3,2}^2 \\ S_{3,3}^2 \end{bmatrix}e^{2i\omega_0 z_0} + \begin{bmatrix} S_{3,1}^3 \\ S_{3,2}^3 \\ S_{3,3}^3 \end{bmatrix}e^{3i\omega_0 z_0} + c.c, \tag{10}$$

Now, the evolution equation can be written as

$$\overrightarrow{S_{3,j}^1} = \begin{pmatrix} f_{1v} + i\omega_0 & f_{2v} & f_{3v} \\ f_{1w} & f_{2w} + i\omega_0 & f_{3w} \\ f_{1z} & f_{2z} & f_{3z} + i\omega_0 \end{pmatrix}\overrightarrow{x_3} \tag{11}$$

From above, this system can be written in a compact form given as

$$\overrightarrow{S_{3,j}^1} = (A - \lambda I)\overrightarrow{x_3} \tag{12}$$

where $\lambda(\mu) = \pm i\omega_0(\mu)$ are the eigenvalues of A. Here, the eigenvalues of A is the function of μ, and variation in μ will cause them to move in the complex plane. If an eigenvalue touch an imaginary axis ($\mathrm{Im}\,\omega_0$), this leads to the case of the fixed point $(3)-(5)$ is no longer hyperbolic.

Setting $r = \dfrac{1}{2} A e^{i\xi}$ and separate equation [1] yields

$$\frac{\partial A}{\partial Z_2} = \frac{S_{1R} A^3}{4} + S_{2R} A \tag{13}$$

where S_{1R} and S_{2R} represent the real part of S_1 and S_2 respectively. Here, the normal form in polar coordinates (A, r) can be written as:

$$\dot{A} = A[S_{2R}(\mu) + \frac{S_{1R}(\mu)}{4} A^2 + S_{3R}(\mu) A^4] = S_{2R} A + \frac{S_{1R}}{4} A^2 + O(A^5) \tag{14}$$

$$\dot{r} = \omega_0(\mu) + O(\mu, A^2) \tag{15}$$

This gives the complex conjugate pair of eigenvalues $S_{2R}(\mu) \pm i\omega_0(\mu)$ and is assumed to satisfy

$$S_{2R}(0) = 0, \omega_0(0) \neq 0 \tag{16}$$

Since only eigenvalues of A on the imaginary axis form a conjugate pair, then the loss of hyperbolicity of Hartman-Grobman Theorem is distinguished.

Let us consider $\dfrac{S_{1R}}{4} < 0$ for example-from (14) the radial equilibria satisfy $A(S_{2R}(\mu) + \dfrac{S_{1R}}{4}(\mu) A^2) \approx 0$ and there are two branches (A, A_H)

$$A = 0 \;,\; A_H = \pm 2 \sqrt{-\frac{S_{2R}}{S_{1R}}} \tag{17}$$

Since A_H must be real, the latter solution is exists only for $S_{2R}(\mu) > 0$. When (15) is taken into account, this new solution in fact describes a periodic orbit of amplitude A_H. Linearize (14) about $A = A_H$ and the linear eigenvalue analytically to check the new solution can be determined. Thus, the system undergoes the condition for stability of Hopf bifurcation A_H is said to be supercritical ($S_{1R} < 0$, $S_{2R} > 0$, $\mu > 0$) or subcritical ($S_{1R} > 0$, $S_{2R} < 0$, $\mu < 0$).

4   UNFOLDING A BIFURCATION

Suppose a stable equilibrium is perturbed by varying an external parameter μ, at a critical value $\mu = \mu_c$, the equilibrium develops a neutral mode. Since at μ_c hyperbolicity is lost, and we must study what happens to the system as μ is varied about μ_c, we give the simplest and most general effect on behaviors of creeping soliton. Here, we investigate the effects on the quantic nonlinear coefficient μ in (1). We vary μ away from critical value μ_C, ($\mu > \mu_c$) which we let at criticality $\mu_c = 0$ for an equilibrium undergoing either steady state or Hopf bifurcation.

The simulation was started from equilibrium $\mu = \mu_c$ up to chaotic solution in the soliton as we attempt to exhibit all possible behaviors of creeping soliton. We fix parameters D, v, ε, β, δ, γ and varies μ.
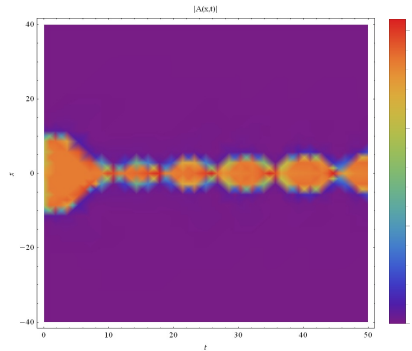
Fig.1. Space-time variation of the $|A|$
for $\delta = -0.1$, $\beta = 0.08$, $D = +1$, $\nu = -0.08$, $\gamma = +1$, $\varepsilon = -0.782$ **and** $\mu = 0.107$.
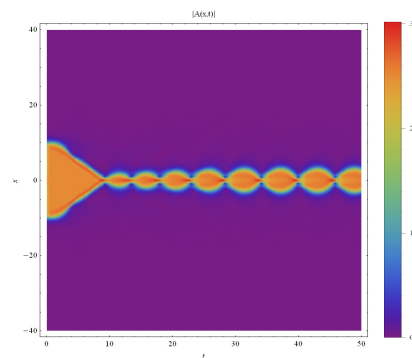

Fig.2. Space-time variation of the wave pattern amplitude $|A|$.
The parameters are $\delta = -0.1$, $\beta = 0.08$, $D = +1$, $\nu = -0.08$, $\gamma = +1$, $\varepsilon = -0.782$ **and** $\mu = 0.121$.

We study the behavior of wave patterns of quantic nonlinear dispersion coefficient of creeping soliton in a spatial domain. We first solved (1) for the set of parameters with an initial condition. Once Fig.1 converges to periodic creeping soliton (Fig.2), we use the initial condition for finding new solutions for different values of parameter as we changed μ while keeping all the parameter fixed. In Fig.3, the solution through a transition when small perturbation occurs when μ increases to 0.165.
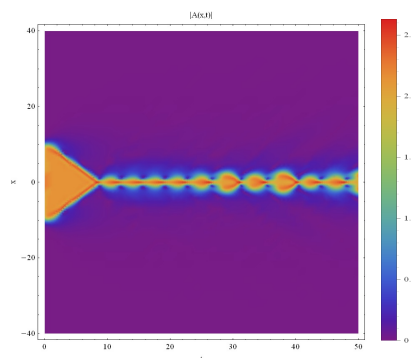

Fig.3. Space-time variation of $|A|$
for $\delta = -0.1$, $\beta = 0.08$, $D = +1$, $\nu = -0.08$, $\gamma = +1$, $\varepsilon = -0.782$ **and** $\mu = 0.165$.

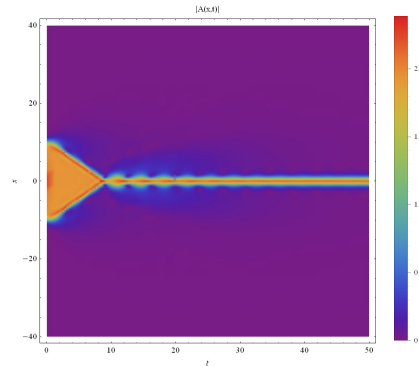**Fig.4. Space-time variation of** $|A|$
for $\delta = -0.1$, $\beta = 0.08$, $D = +1$, $v = -0.08$, $\gamma = +1$, $\varepsilon = -0.782$ **and** $\mu = 0.20$.

The variation of the coefficient μ leads to bifurcation of equilibrium solution to chaotic states, which is illustrated in Fig.1 until Fig.4. These figures show the nature of the transitions can occur if we changed one parameter as a control bifurcation coefficient. It reveals that the effects on the pattern from vary the quintic nonlinear parameter μ are propagate in only one direction toward the left and decreasing in time. We were able to find out how the solutions change their behavior as μ changed and we have noticed that when μ further increases, the chaotic motion frequently spread the soliton travels before transforming each pulsations getting closer. As we discovered the behavior of the pattern and the disordered regimes in the domain become more complex, the system becomes more and more chaotic (Fig.4).

5   CONCLUSION

In this paper, we have studied the behavior of quintic nonlinear dispersion coefficient of creeping solitons using variational approach as we relate our work with Hartman-Grobman Theorem. Other than that, we have discussed briefly on the theoretical work for analyzing the hyperbolicity of one class of solitary waves (creeping soliton) in the CGLE and presented some numerical simulations for dissipative creeping soliton solutions to the CGLE. The dynamics of the wave patterns were controlled by one parameter, μ in (1). We attempted to exhibit all possible behaviors for the system of equilibrium, $\mu = \mu_C$, up to chaotic solution in creeping solitons. Since μ is driven out the oscillatory instability exceeded the critically parameter which is corresponding to the Hopf bifurcations as the system is highly complex.

We found that the regimes are disordered in the domain and equilibrium states undergo propagation of the chaotic states when μ increases.

In the case of hyperbolic analysis, the eigenvalues of Jacobian matrix touch the imaginary axis. This leads to hyperbolicity loss of Hartman-Grobman Theorem in the dissipative system.

6   ACKNOWLEDGMENT

7   REFERENCES

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