# AN IMPLICIT FINITE DIFFERENCE METHOD FOR A FORCED KDV EQUATION

L.H. Wiryanto[1] and Achirul A.[2]
[1, 2]Department of Mathematics
Bandung Institute of Technology
Jalan Ganesha 10 Bandung-Indonesia
e-mail : leo@math.itb.ac.id

**Abstract.** A finite difference method is developed to solve a forced KdV equation representing a surface elevation of fluid flowing on a channel with a small bump at the bottom. We indicate some difficulties in solving the equation since it has a nonlinear and third derivative terms. We present the technique in this paper to solve the equation. As the result, the numerical scheme gives solutions performing nonlinear wave-trains of water surface generated by the forcing term.

**Keywords:** KdV equation, Froude number

## 1. Introduction

When fluid flow is disturbed by a small bump, it can generate surface wave. For an ideal fluid and irrotational flow, the wave generation can be modeled in a non-homogenous KdV equation of the surface elevation that is called forced KdV equation, i.e. KdV equation with a forcing term. Cole [1] derived the equation for Froude numbers sufficiently close to 1, with which the non-homogenous term relates to the profile of the bump. The equation is then integrated and is discritized to get an implicit numerical scheme. As the boundaries, the elevation is fixed at zero. In this numerical procedure, Cole [1] obtained nonlinear wave-trains running upstream and downstream from the bump. Grimshaw et. al. [3] showed that the wave-trains having the structure of unsteady undular bores analytically and also confirmed by numerical solutions using an explicit finite difference method.

In this paper, we consider non-dimensional fKdV equation of water surface elevation $u(x,t)$ satisfying

$$2u_t + Ku_x - 3uu_x - \frac{1}{3}u_{xxx} = f_x(x) \qquad (1)$$

as the flow is disturbed by a bump $f(x)$ for a small interval, derived by Cole [1]. Here, $K$ is the correction factor of small order $\varepsilon^{3/2}$ to the Froude number $F$, as defined to the undisturbed water depth, that is $F^2 = 1 + \varepsilon^{3/2}K$. The equation is solved numerically by an implicit finite difference method as developed by Feng and Mitsui [2] and applied to a KdV equation of interface wave by Wiryanto and Djohan [4]. The method is able to perform wavetrains as obtained by Grimshaw et. al. [3], and then used to observe wave generation for various disturbances $f(x)$.

In the following sections, the numerical method is described in section 2. Scaling and differentiating are applied to the equation before discritizing into a system of linear equations. This step is required to observe the effect of the nonlinear term and to obtain the consistent system by constructing a matrix having diagonal dominant. In section 3 numerical results are presented.

## 2. Numerical Method.

The nonlinear equation (1) is considered to be solved numerically, by providing its initial condition $u(x,0)$ and boundary condition $u(x,t) \to 0$ as $|x| \to \infty$. Following Wiryanto and Djohan [4], the equation (1) is first differentiated to $x$, giving

$$2u_{tx}+Ku_{xx}-\frac{3}{2}\left(u^2\right)_{xx}-\frac{1}{3}u_{xxxx}=f_{xx}(x),$$

since the term having third derivative producing un-balances order to others. Equation (2) is then discritized by forward-time central-space. A system of linear equations is constructed by involving two-level average for derivative terms with respect to $x$, i.e.

$$\left.\begin{aligned} u_{tx} &\approx \frac{1}{2\Delta t\Delta x}\left(u_{i+1}^{j+1}-u_{i-1}^{j+1}-u_{i+1}^{j}+u_{i-1}^{j}\right)\\ u_{xx} &\approx \frac{1}{2\Delta x^2}\left(u_{i-1}^{j+1}-2u_{i}^{j+1}+u_{i+1}^{j+1}\right.\\ &\qquad \left.+u_{i-1}^{j}-2u_{i}^{j}+u_{i+1}^{j}\right)\\ u_{xxxx} &\approx \frac{1}{2\Delta x^4}\left(u_{i-2}^{j+1}-4u_{i-1}^{j+1}+6u_{i}^{j+1}-4u_{i+1}^{j+!}\right.\\ &\qquad +u_{i+2}^{j+1}+u_{i-2}^{j}-4u_{i-1}^{j}+6u_{i}^{j}\\ &\qquad \left.-4u_{i+1}^{j}+u_{i+2}^{j}\right) \end{aligned}\right\} \tag{3}$$

where $\Delta x$ and $\Delta t$ are the length of the space and time discritisation, and we denote $x=i\Delta x, t=j\Delta t$ for $i=0,1,2,\cdots,N$ and $j=0,1,2,\cdots,M$ for the discritizing of space and time domains, so that we then denote $u(x,t)=u_i^j$. The nonlinear term is approximated by applying the first order of Taylor series. We suppose $g=u^2$. Therefore, the second derivative $g_{xx}$ is discritized into

$$g_{xx}\approx\frac{1}{2\Delta x^2}\left(g_{i-1}^{j+1}-2g_{i}^{j+1}+g_{i+1}^{j+1}+g_{i-1}^{j}-2g_{i}^{j}+g_{i+1}^{j}\right) \tag{4}$$

Meanwhile, from truncated Taylor series

$$g_i^{j+1}\approx g_i^j+g_u u_t\big|_i^j\Delta t \tag{5}$$

the two-level average of $g$ can be approximated by

$$\frac{1}{2}\left(g_i^{j+1}+g_i^j\right)\approx u_i^j u_i^{j+1} \tag{6}$$

This is then applied to (4) giving (2)

$$g_{xx}\approx\frac{1}{\Delta x^2}\left(u_{i-1}^{j+1}u_{i-1}^{j}-2u_{i}^{j+1}u_{i}^{j}+u_{i+1}^{j+1}u_{i+1}^{j}\right) \tag{7}$$

as the liniarizing of the third term of (2). Now the finite different equation of Equation (2) can be obtained by using (3) and (7). The external force, the right hand side of (2), can be discritized after differentiating $f$ or by approximating $f_{xx}$ from given $f$. A set of linear equations is obtained that can be written in form of

$$\left.\begin{aligned} a_0u_{i-2}^{j+1}+a_1u_{i-1}^{j+1}+a_2u_{i}^{j+1}+a_3u_{i+1}^{j+1}&\\ +a_0u_{i+2}^{j+1}=b& \end{aligned}\right\} \tag{8}$$

for $i=0,1,2,\cdots,N$ at time step $j+1$; and the coefficients are

$$a_0=-\frac{\Delta t}{6\Delta x^3}$$

$$a_1=-1+\frac{\Delta t}{4\Delta x}\left(K-3u_{i-1}^j\right)+\frac{2\Delta t}{3\Delta x^3}$$

$$a_2=\frac{\Delta t}{\Delta x}\left(-K+3u_i^j\right)-\frac{\Delta t}{\Delta x^3}$$

$$a_3=1+\frac{\Delta t}{4\Delta x}\left(K-3u_{i+1}^j\right)+\frac{2\Delta t}{3\Delta x^3}$$

Meanwhile, the right hand side of (8) is a constant obtained from previous time step, i.e.

$$b=2\Delta t\Delta x f_{xx}-a_0u_{i-2}^j-b_1u_{i-1}^j-b_2u_i^j-b_3u_{i+1}^j-a_0u_{i+2}^j \tag{9}$$

where

$$b_1=1+\frac{K\Delta t}{2\Delta x}+\frac{2\Delta t}{3\Delta x^3}$$

$$b_2=-\frac{K\Delta t}{\Delta x}-\frac{\Delta t}{\Delta x^3}$$

$$b_3=1+\frac{K\Delta t}{2\Delta x}+\frac{2\Delta t}{3\Delta x^3}$$

The equations (8) become a closed system by involving left and right boundaries $u_{-2}^{j+1},u_{-1}^{j+1}$ and $u_{N+1}^{j+1},u_{N+2}^{j+1}$ for each $j$. We can give zero, representing undisturbed surface at both boundaries, by assuming that those places are far enough from bump. Therefore, the numerical procedure has to be stopped when the generated wave reaches both boundaries. As the initial condition, various functions can be used such as $u(x,0)=0$ representing no wave, or a solitary wave in form of each function. The closed system is then solved numerically by Gauss-Seidel method or other methods for each time step.

## 3. Numerical Results.

The numerical procedure described in the previous section is used to observe waves generated by flow disturbed by a bump. Most of our calculations use $\Delta x = 0.02, \Delta t = 0.01$ and number of time step $M = 10000$ for observation domain in $x$ $[0, 200]$. The bump is placed in the middle of the domain.

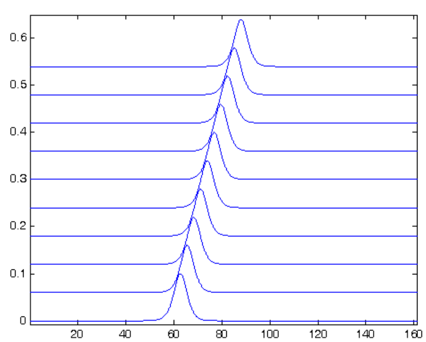



Figure 1: Plot of $u(x,t)$ for some of $t$ calculated from KDV equation of (1) using $K = 1.5$.

The scheme is firstly tested for KdV equation by giving the force $f(x)=0$ in the right hand side of (2), and the initial condition is

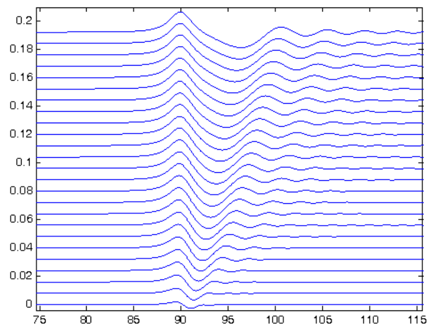
$$u(x,0) = a\,\mathrm{sech}^2 b\left(x - x_0\right) \qquad (10)$$

with the crest located at $x = x_0$. Analitically, the initial condition (10) produces a solitary wave traveling with constant speed and without changing the shape when $b = \sqrt{3a/4}$ for the KdV equation of (1). These characters are demonstrated in Figure 1 as a solution of KdV equation of (1) for $K = 1.5$, and the initial condition is given in (10) for $a = 0.1$, $x_0 = 60$. Plot of $u(x,t)$ for some values $t$ is shifted upward proportional to the value of $t$. For smaller values $K$ the wave travels to the right slower, and if it is continued decreasing $K$ the wave travels in the different direction. At $K = a$ the wave reaches steady. This agrees with analytical solution, i.e.

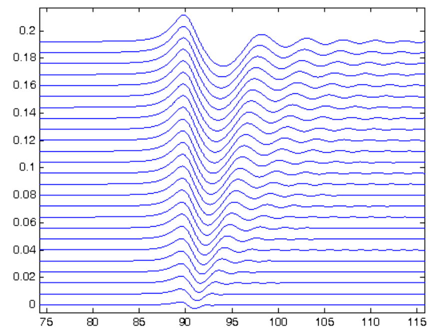
$$u(x,t) = a\,\mathrm{sech}^2 \sqrt{\frac{3a}{4}}\left[\left(x - x_0\right) - \frac{K-a}{2}t\right] \qquad (11)$$

Now the scheme is used to solve (1) involving the forcing term. The parameter $K$ and the function $f$ are the input considered effecting to the solution. First, we perform the solution for various values of $K$ combined with a bump $f(x) = 0.02\,\mathrm{sech}^2\left[0.12\left(x-90\right)\right] - 0.0319$ for $x \in [84, 96]$ and $f(x) = 0$ for other. We show plot $u(x,t)$ in Figure 2 corresponding to $K = 1.0, 0.5, 0.02$, and $-1.0$. Two points are indicated as the place where the wave appears, i.e. at the left and right ends of the bump, $x = 84$ and $x = 96$. The surface above the left end is pushed up and the other end is opposite direction, as below the flow is disturbed and then runs down freely after passing the bump. This continuously grows up the elevation and followed by appearing other waves. Different value of $K$ can be seen the evolution of both waves in figure 2. It is indicated that the elevation above $x = 84$ and $x = 96$ grows up without much propagating or generating other waves when $K$ tends to the amplitude of the bump. We can see this in figure 2c for the same value of $K$ and the amplitude.

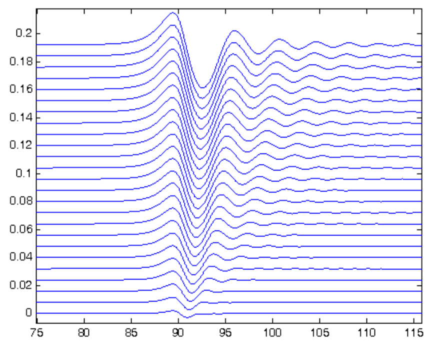
We then observe the effect of the forcing term by replacing the secan-hyperbolic function above with other function such as sinusoidal. We obtain similar profile of the elevation. But, when



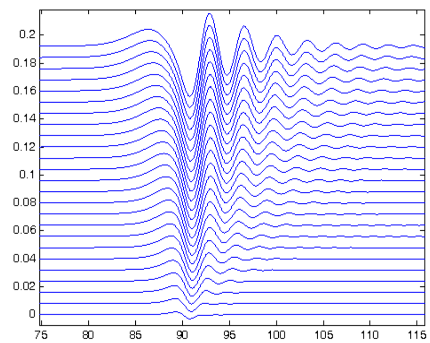

(a) $K = 1.0$




(b) $K = 0.5$




(c) $K = 0.02$




(d) $K = -1.0$

Figure 2: Plot of $u(x,t)$ for (a) $K = 1.0$, (b) $0.5$, (c) $0.02$, (d) $-1.0$, with the bump as given in the text.

the forcing term is negative, we obtain opposite composition of the surface elevation described above, the left end of the negative bump producing negative elevation and the right end of the bump producing positive one.

**Conclusions**

We have developed an implicit finite difference method to solve a forced KdV equation representing wave generation as a uniform flow disturbed by a bump on the bottom of a channel. The method has been tested by comparing the numerical solution of KdV equation to the analytical solution in form of solitary wave, and we obtain agreement between both solutions. When a forcing term is included we obtain that the solution describes generating a train of waves such as obtained by Cole [1]. The evolution of the waves is observed for various parameter corresponding to the Froude number.

**Acknowledgement.** The research of this paper was partly supported by Research Grant from Bandung Institute of Technology, contract number 004/K01.03.2/PL2.1.5/I/2006, and this paper is part of the final project of the second Author.

**References**

[1] S. L. Cole (1985), *Transient waves produced by flow past a bump*. Wave Motion, Vol. 7, 579-587.

[2] B –F. Feng and T. Mitsui (1998), *A finite Difference Method for KdV and KP equations*, J. Comp. Applied Maths., Vol. 90, 95-116.

[3] R. H. J. Grimshaw and D.–H. Zhang. *Generation of Solitary Waves by Transcritical Flow Over A Step*. www. lboro.ac. uk/departments/ma/research/ preprints/papers07/07-02.pdf

[4] L.H. Wiryanto and Warsoma Djohan (2006), *Metoda Beda Hingga pada Persamaan KdV Gelombang Interface*, Jurnal Matematika, Vol. 9, 117-123.